Modulo Quantization Coding for Gaussian Primitive Diamond Channel with Correlated Noises

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***Abstract*—This paper proposes a modulo quantization (MQ) coding strategy for the Gaussian primitive diamond channel with perfect or near-perfect noise correlation at the relay nodes. In a diamond channel, the two relays observe noise-corrupted versions of the input signal, and have rate-limited noiseless forward links to a central receiver that does not have direct observation of the input. MQ is a structured operation consisting of scalar quantization of a signal followed by a modulo operation. For the perfect noise correlation case, the algebraic properties of MQ allow it to exploit the channel determinism induced by the noise correlation. We show that it is possible to achieve an overall rate equal to the minimum of the two forward relay-link capacities for the diamond channel with perfectly correlated noises, using only arbitrarily small non-zero input power. Furthermore, we propose a hybrid scheme combining MQ and decode-forward (DF) that yields a new achievable rate, and when the relay-link capacities are unequal, achieves the capacity of the diamond channel in certain low signal-to-noise ratio (SNR) regimes. For the near-perfect noise correlation case, MQ can still outperform conventional relay strategies such as DF and compress-forward (CF) at low SNR.**

I. Introduction

When a communication channel exhibits special structure, coding schemes that exploit such structure can often outperform conventional strategies that ignore it. This paper shows that modulo quantization (MQ), a simple structured operation, provides an effective coding approach for the Gaussian primitive diamond channel with strongly correlated noises.

The diamond channel refers to a channel model with a transmitter, two relays, and a receiver. It consists of a broadcast channel (BC) as the first hop in which noisy versions of the transmit signal are observed at the two relays, and a multiple-access channel (MAC) as a second hop, in which the relays subsequently communicate with the receiver to help it decode the message from the transmitter. The diamond channel has been studied extensively in the literature [1]–[11], but its capacity characterization is still largely open. One of the main difficulties in deriving capacity results for the diamond channel is that the observations at the relays are correlated due to the broadcast nature of the first hop, but it is not clear how to best take advantage of such correlation in the second hop. While conventional relay strategies such as decode-and-forward (DF) and compress-and-forward (CF) can be used to derive achievable rates [1], [2], capacity results for the diamond channel are available only for some special cases, for example, when one of the relays observes a deterministic function of the channel input [3], [4], or certain cases when the BC component is digital [5], [6], [8].

In this paper, we focus on the *primitive* diamond relay channel, in which the two relay nodes have noiseless finite-capacity links to the receiver. In addition, we restrict attention to the case in which the BC component is Gaussian and consider the setting in which the correlation of the relay observations is due not only to the common transmit signal, but also to the *correlated Gaussian noises*. We show that when the noises are perfectly correlated, they induce certain channel determinism that allows a novel structured MQ coding strategy to be developed. Such a structured coding scheme can lead to new achievability and even capacity results.

The diamond relay channel with correlated noises has been studied recently in [12], [13], where achievability bounds based on DF and CF relay strategies have been derived. This paper differs from [12], [13] in that we take advantage of the following crucial observation in designing the relay strategies. When the noises are perfectly correlated, there is a deterministic relationship among the input and the two relay observations in the sense that any one of the three random variables can be expressed as a function of the other two.

Determinism of this kind has previously enabled capacity characterizations for several related channels, including the primitive relay channel [14] and the BC with conferencing decoders [15]. In this paper, we show that such determinism can be effectively exploited by MQ coding, originally developed in [16]. Furthermore, by combining MQ coding with DF, a hybrid MQ–DF scheme can improve upon previously known achievable rates for the diamond channel [12], [13]. When the relay links have different capacities, hybrid MQ-DF gives rise to capacity results in certain low signal-to-noise ratio (SNR) regimes. This paper further extends the MQ framework to the case of near-perfect noise correlation, and shows that the MQ-based scheme can still outperform DF and CF at low SNR.

The notations used in the paper are as follows. The set of positive integers is denoted $\mathbb{Z}_+$. For $a \in \mathbb{Z}_+$, we use $[a]$ to denote the set $\{0, 1, \cdots, a-1\}$. The Gaussian distribution with mean μ and variance σ^2 is denoted by $\mathcal{N}(\mu, \sigma^2)$. The shorthand $\psi(\cdot)$ is used for the Gaussian capacity function $\psi(x) \triangleq \frac{1}{2}\log(1+x)$. All logarithms are of base 2.

II. System Model and Preliminaries

A. Gaussian Primitive Diamond Channel

The Gaussian primitive diamond channel is depicted in Fig. 1. Given input X, the two relays observe $Y_1 = \alpha X + Z_1$ and $Y_2 = \beta X + Z_2$ respectively, where α, β are the corresponding channel gains, and Z_1 and Z_2 are Gaussian random

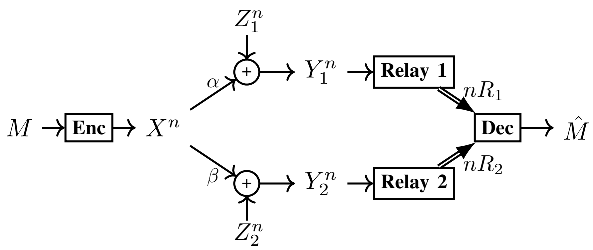


Fig. 1. The Gaussian primitive diamond channel.

variables each with unit variance. We consider two cases: the perfect noise correlation case in which the relays share a common noise variable, $Z_1 = Z_2 \triangleq Z$; and non-perfect correlation in which the noise correlation coefficient is denoted by $\lambda \triangleq \mathbb{E}[Z_1 Z_2]$. The two relays communicate with a central receiver through primitive relay links of capacities R_1 and R_2 bits per channel use, respectively. Importantly, there is no direct link from the transmitter to the central receiver; the receiver obtains its information exclusively through the relays.

A $(2^{nR}, n)$-code for this channel is specified by the quadruple $(f_{\mathsf{enc}}^{(n)}, f_{\mathsf{relay,1}}^{(n)}, f_{\mathsf{relay,2}}^{(n)}, f_{\mathsf{dec}}^{(n)})$. The encoding function $f_{\mathsf{enc}}^{(n)} : [2^{nR}] \to \mathbb{R}^n$ maps a random message M to a codeword $X^n = x^n(M) \triangleq f_{\mathsf{enc}}^{(n)}(M)$ such that the average power constraint $\frac{1}{n}\|x^n(m)\|^2 \le P$ is satisfied for all $m \in [2^{nR}]$. For $j = 1, 2$, the j-th relay forwards the nR_j-bit description of its observation $V_j = f_{\mathsf{relay,j}}^{(n)}(Y_j^n)$ to the receiver through a noiseless link, where $f_{\mathsf{relay,j}}^{(n)} : \mathbb{R}^n \to [2^{nR_j}]$ is the corresponding relay function. The decoding function $f_{\mathsf{dec}}^{(n)} : [2^{nR_1}] \times [2^{nR_2}] \to [2^{nR}]$ for this channel now outputs the estimated message $\hat{M} \triangleq f_{\mathsf{dec}}^{(n)}(V_1, V_2)$. A rate R is achievable if for every $\epsilon > 0$, there exists a sequence of $(2^{nR}, n)$-codes such that for sufficiently large n, the error probability $P_e^{(n)} \triangleq \Pr\{\hat{M} \neq M\}$ is upper bounded by ϵ. The capacity of this primitive relay channel $C(R_1, R_2; \lambda)$ is the supremum of all achievable rates. For the perfect noise correlation case, we denote the capacity simply by $C(R_1, R_2)$.

B. Modulo Quantization

Modulo quantization (MQ) is a simple operation consisting of a uniform scalar quantization followed by a modulo operation. While similar operations appear earlier as ad-hoc constructions (e.g., in [17]), MQ is developed formally as a systematic coding technique in [16], where its algebraic properties are exploited for structured coding in the context of the primitive relay channel. In the following, we define the operation and state the properties relevant to the present work.

Fix *quantization step* $\Delta > 0$ and *modulo parameter* $k \in \mathbb{Z}_+$. The (Δ, k)-*modulo quantization* is the function $\lceil \cdot \rfloor_{\Delta,k} : \mathbb{R} \to [k]$ defined as

$$\lceil x \rfloor_{\Delta,k} \triangleq \left\lfloor \frac{x}{\Delta} \right\rfloor \bmod k, \tag{1}$$

where $\lfloor \cdot \rfloor$ denotes the largest integer less than or equal to its argument, and $(\cdot) \bmod k$ denotes the remainder upon dividing its argument by k.

The key structural property of MQ is an identity relating MQ outputs whose inputs differ by an integer multiple of the quantization step. The formal statement is presented in the lemma below.

Lemma 1. *Fix any $\Delta > 0$ and $k \in \mathbb{Z}_+$, the following identity holds for all $x \in \mathbb{R}$ and $a \in [k]$:*

$$(\lceil x + a\Delta \rfloor_{\Delta,k} - \lceil x \rfloor_{\Delta,k}) \bmod k = a. \tag{2}$$

The derivation of the lemma follows directly from the definition of the MQ operation.

III. Perfect Noise Correlation

In this section, we propose MQ-based coding strategies for the Gaussian primitive diamond channel with perfect noise correlation. We exclude the special case in which the relay channel gains are identical, i.e. $\alpha = \beta$. In this degenerate setting, the two distributed relays are equivalent to a single relay with observation $\alpha X + Z$ and an outgoing bit pipe of capacity $R_1 + R_2$, for which the capacity is simply the minimum of the capacities of two hops.

For the non-degenerate $\alpha \neq \beta$ case, we first establish a coding scheme that relies on the structural properties of MQ to exploit the noise correlation. Moreover, we propose a way to combine this MQ scheme with DF to yield a new achievability bound for the diamond channel. Finally, we show that this new bound is tight in certain parameter regimes, thereby revealing a new capacity result.

A. MQ Coding

We now present MQ coding as a simple, explicit, low-complexity scalar scheme that achieves *zero error* for transmitting $\lfloor 2^{R_{\min}} \rfloor$ messages in a single channel use using an *arbitrarily small* input power, where $R_{\min} \triangleq \min\{R_1, R_2\}$.

Proposition 1. *Assume $\alpha \neq \beta$. For the Gaussian primitive diamond channel with perfectly correlated noises at the relays, there exists a $(\lfloor 2^{R_{\min}} \rfloor, 1)$ code with zero error probability for any positive average power constraint $P > 0$.*

Proof: The MQ coding scheme is described as follows. Let the message set size be $K = \lfloor 2^{R_{\min}} \rfloor$. Set $\Delta = |\alpha - \beta|\sqrt{P}/K$. For a random message $M \in [K]$, the transmitted signal X is set to be

$$X = f_{\mathsf{enc}}(M) = M\Delta/(\alpha - \beta) \tag{3}$$

The power constraint is satisfied, because for all $m \in [K]$

$$|f_{\mathsf{enc}}(m)|^2 = \left| \frac{m}{\alpha - \beta} \cdot \frac{|\alpha - \beta|\sqrt{P}}{K} \right|^2 \le P. \tag{4}$$

Each of the relays forwards the (Δ, K)-modulo quantization output of its observation to the receiver:

$$V_i = f_{\mathsf{relay,i}}(Y_i) = \lceil Y_i \rfloor_{\Delta,K}. \tag{5}$$

Note that since $K \le \lfloor 2^{R_i} \rfloor$ for $i = 1, 2$, forwarding V_i to the destination does not exceed the capacity of the link between

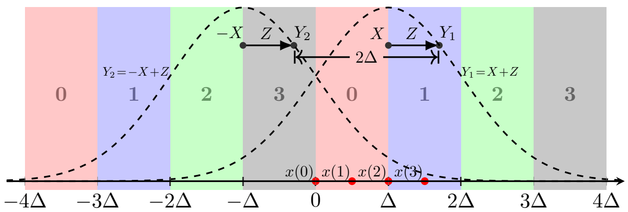


Fig. 2. Illustration of the MQ coding scheme. In this example, the channel parameters are $\alpha = -\beta = 1$ and $R_{\text{min}} = 2$, and the message set size is $K = \lfloor 2^{R_{\text{min}}} \rfloor = 4$. Message $M = 2$ is transmitted. The transmit signal is $X = x(2) = \Delta$. The received signals Y_1 and Y_2 always differ by 2Δ. The relaying strategy is to forward $V_1 = 1$ and $V_2 = 3$. The decoded message is $\hat{M} = (V_1 - V_2) \bmod K = (1-3) \bmod 4 = 2$, which equals M.

the i-th relay and the destination. The central receiver computes the estimate

$$\hat{M} = f_{\mathsf{dec}}(V_1, V_2) = (V_1 - V_2) \bmod K. \tag{6}$$

We now show that $\hat{M}$ is always equal to M. Observe that

$$Y_1 = \alpha X + Z = (\beta X + Z) + (\alpha - \beta)X = Y_2 + M\Delta, \tag{7}$$

where the last equality is due to (3). We then obtain that

$$\begin{aligned} \hat{M} &= (V_1 - V_2) \bmod K \\ &= (\lceil Y_2 + M\Delta \rfloor_{\Delta,K} - \lceil Y_2 \rfloor_{\Delta,K}) \bmod K = M, \end{aligned} \tag{8}$$

where the last equality is due to Lemma 1. ∎

To understand how the MQ coding is able to exploit perfect noise correlation, despite having only rate-limited relay links to the receiver, we provide a graphical illustration for the case of $\alpha = -\beta = 1$ and $R_{\text{min}} = 2$ in Fig. 2. For fixed Δ, the real line is partitioned into $K = 4$ different regions, indicated by different colours (red, blue, green, grey in the figure). Each region corresponds to a message $m \in \{0, 1, 2, 3\}$ under the (Δ, K)-modulo quantization operation and is a union of countably many length-Δ intervals. Suppose the message $M = m$ is transmitted, then the relay observations Y_1 and Y_2 always differ by exactly $m\Delta$, regardless of the realization of the noise Z. Consequently, taking the modulo-K difference of the (Δ, K)-modulo-quantized relay outputs always recovers m perfectly. This ensures error-free decoding, while satisfying the relay-destination link capacity $R_{\text{min}} = \log K$. Importantly, this argument holds for any choice of $\Delta > 0$, and hence the MQ scheme can operate with arbitrarily small transmit power by choosing Δ sufficiently small. This strategy takes advantage of the presence of certain determinism in the channel, namely, given any two of three random variables (X, Y_1, Y_2), the other random variable can be found deterministically.

Proposition 1 states that any rate below R_{min} can be achieved using MQ coding with *arbitrarily small* (yet non-zero) power in a *single* time slot. In the next section we build on this result when combining MQ with other relay strategies.

B. Hybrid MQ-DF Coding Scheme

When the relay-link capacities are unequal, MQ coding can only use R_{min} bits per channel use of the higher-capacity relay link. This leaves the balance of the link capacity unused. Further, the MQ rate is independent of the input power, and hence cannot leverage higher operating SNRs.

The aforementioned limitations motivate the following length-n block hybrid scheme that combines MQ and DF. We partition the block into one single channel use for MQ and $n-1$ channel uses for DF. We use a constant transmit power P in each channel use, thus the total energy nP is also partitioned, so that P is allocated to MQ and $(n-1)P$ is allocated to DF. Finally, we partition the relay-link capacities such that each relay uses $n\rho \in [0, nR_{\text{min}}]$ bits for MQ and the rest of the excess capacity is used for DF. These choices are made because MQ only requires one channel use with arbitrarily small power to achieve a rate equal to the minimum of two relay link rates. Here, we do not necessarily allocate the full nR_{min} to MQ, because it may be advantageous to leave some of the relay link rates to the DF phase. In the DF phase, we further use parameter $\gamma \in [0, 1]$ to control the power allocation between the private and common messages.

Per the discussion above, the MQ phase of the proposed scheme takes place only in the first channel use. In this single channel use, each relay sends $n\rho$ bits to the receiver. According to Proposition 1, $\log\lfloor 2^{n\rho} \rfloor = n(\rho - o(1))$ bits of information can be conveyed at zero error using any non-zero power.

For the remaining $(n-1)$ channel uses, we employ a DF scheme with power P per channel use. Note that the Gaussian primitive diamond channel can be regarded as a degraded Gaussian BC cascaded with a noiseless digital MAC. To apply DF, we use superposition coding with a common and a private message for the Gaussian BC component. The private message is encoded using a Gaussian codebook with i.i.d. $\mathcal{N}(0, \gamma P)$ entries. The common message is encoded using a Gaussian codebook with i.i.d. $\mathcal{N}(0, (1-\gamma)P)$ entries. Both relays decode the common message. Only the stronger relay (the relay with the larger magnitude channel gain) decodes the private message. The stronger relay forwards its private message; any residual capacity of its relay link is used to forward part of the common message. The weaker relay forwards the balance of the common message.

The receiver decodes the MQ message at zero-error, and obtains the DF messages from the digital links. The following theorem gives the rate achieved using this scheme.

Theorem 1. *Assume $|\alpha| \geq |\beta|$ and $\alpha \neq \beta$. The capacity of the Gaussian primitive diamond channel with perfectly correlated noises at the relays is lower bounded by*

$$\begin{aligned} &R_{\text{MQ-DF}}(R_1, R_2) \\ &\triangleq \max_{\substack{\gamma \in [0,1] \\ \rho \in [0, R_{\text{min}}]}} \min \left\{ \begin{matrix} R_1 + R_2 - \rho \\ R_1 + \psi\left(\frac{(1-\gamma)\beta^2 P}{\gamma\beta^2 P + 1}\right) \\ \psi(\gamma\alpha^2 P) + \psi\left(\frac{(1-\gamma)\beta^2 P}{\gamma\beta^2 P + 1}\right) + \rho \end{matrix} \right\}. \end{aligned} \tag{9}$$

Proof: Fix γ, ρ. Denote the common and private message rate (normalized by n) of the DF component by $R_{\text{DF-comm}}$ and $R_{\text{DF-priv}}$ respectively. As $n \to \infty$, the superposition coding inner bound of BC gives

$$R_{\text{DF-comm}} \leq \psi\left(\frac{(1-\gamma)\beta^2 P}{\gamma\beta^2 P + 1}\right), \ R_{\text{DF-priv}} \leq \psi(\gamma\alpha^2 P). \tag{10}$$

Meanwhile, in order to forward the MQ message and the DF messages to the receiver via the relay links, the following bounds hold as $n \to \infty$:

$$R_{\text{DF-priv}} + \rho \le R_1, \tag{11}$$

$$R_{\text{DF-priv}} + R_{\text{DF-comm}} + 2\rho \le R_1 + R_2. \tag{12}$$

The total message rate for this scheme is $\rho + R_{\text{DF-priv}} + R_{\text{DF-comm}}$. By eliminating the parameters $R'_{\text{DF-comm}}$ and $R'_{\text{DF-priv}}$ via Fourier-Motzkin elimination, and further optimizing the parameters γ and ρ, we conclude that the rate expression in (9) is achievable. ■

C. Capacity Characterization

The MQ-DF achievability bound (9) turns out to achieve the cut-set bound in certain parameter regimes. For general diamond networks, the cut-set bounds remain the best known converse [18, Chapter 15.1]. The bound can be expressed as the minimum of four terms: a broadcast cut-set, a multiple-access cut-set, and two cross cut-sets [1, Chapter 3.2]. For the Gaussian primitive diamond channel with perfect noise correlation, the channel input can be written as a function of the relay outputs. This makes the broadcast cut-set vacuous. As shown in the following proposition, this observation allows the cut-set bound to be simplified.

Proposition 2. *Assume $\alpha \neq \beta$. The capacity of the Gaussian primitive diamond channel with perfectly correlated noises is bounded from above by the cut-set bound:*

$$\text{Cutset}(R_1, R_2) = \min \left\{ \begin{matrix} R_1 + R_2 \\ R_1 + \psi(\beta^2 P) \\ R_2 + \psi(\alpha^2 P) \end{matrix} \right\}. \tag{13}$$

By comparing the rate achieved in (9) with the converse bound (13), we observe that the two bounds coincide at high SNR. Whenever the two relay-link capacities differ, the bounds also coincide at low SNR. The following theorem formalizes this observation.

Theorem 2. *Assume $\alpha \neq \beta$ and $|\alpha| \ge |\beta|$. The capacity of the Gaussian primitive diamond channel with perfectly correlated noises is characterized in the following cases:*

(a) For $P \ge \frac{2^{2R_2}-1}{\beta^2} + \frac{2^{2R_2}(2^{2R_1}-1)}{\alpha^2}$,

$$C(R_1, R_2) = R_1 + R_2. \tag{14}$$

(b) If $R_1 > R_2$, then for $P \le \frac{2^{2(R_1-R_2)}-1}{\alpha^2}$,

$$C(R_1, R_2) = R_2 + \psi(\alpha^2 P). \tag{15}$$

If $R_2 > R_1$, then for $P \le \frac{2^{2(R_2-R_1)}-1}{\beta^2}$,

$$C(R_1, R_2) = R_1 + \psi(\beta^2 P). \tag{16}$$

We now explain why the MQ-DF scheme can achieve capacity on this diamond channel with perfectly correlated noises in the two different regimes. In the high-SNR regime, the optimal rate-allocation parameter in (9) is $\rho^* = 0$. In this regime, MQ is not used, and the hybrid scheme reduces to DF. This is capacity achieving, because when the input power P is sufficiently large, each relay can decode the transmitted message at a rate equal to its outgoing bit-pipe capacity. This saturates the multiple-access cut-set bound.

In the more interesting low-SNR regime, the optimum ρ is $\rho^* = R_{\min}$. This means that the MQ component operates at its largest permissible rate. The DF component becomes degenerate, as it now operates on the two-hop relay channel involving only the relay with higher forward link capacity with outgoing link capacity $R_{\text{ex}} \triangleq |R_1 - R_2|$. In this low-SNR regime, the capacity of this two-hop relay channel is given by the capacity of the first hop, which is a single-user Gaussian channel with power P and channel gain α or β, depending on which relay has excess forward link capacity.

We note that in the symmetric relay-link capacity case $R_1 = R_2 \triangleq \bar{R}$, the capacity result reduces to $C(\bar{R}, \bar{R}) = \bar{R}$ asymptotically as $P \to 0$. This is achieved with MQ alone.

D. Comparison of Bounds on Capacity

To better understand the behavior of the MQ-DF achievability bound (9), we now numerically compare it with other known bounds on the capacity of the Gaussian primitive diamond channel with perfect noise correlation.

For converse, we plot the cut-set bound (13). For achievability, we plot the DF and CF achievability bounds as benchmarks for comparison. The pure DF scheme can be seen as a special case of MQ-DF with the rate-allocation parameter fixed to $\rho = 0$. As a result, the DF achievability bound admits an optimization formulation similar to (9), and simplifies in the symmetric case as

$$R_{\text{DF}}(R_1, R_2) \triangleq \max_{\gamma \in [0,1]} \min \left\{ \begin{matrix} R_1 + R_2 \\ R_1 + \psi\left(\frac{(1-\gamma)\beta^2 P}{\gamma\beta^2 P + 1}\right) \\ \psi(\gamma\alpha^2 P) + \psi\left(\frac{(1-\gamma)\beta^2 P}{\gamma\beta^2 P + 1}\right) \end{matrix} \right\}. \tag{17}$$

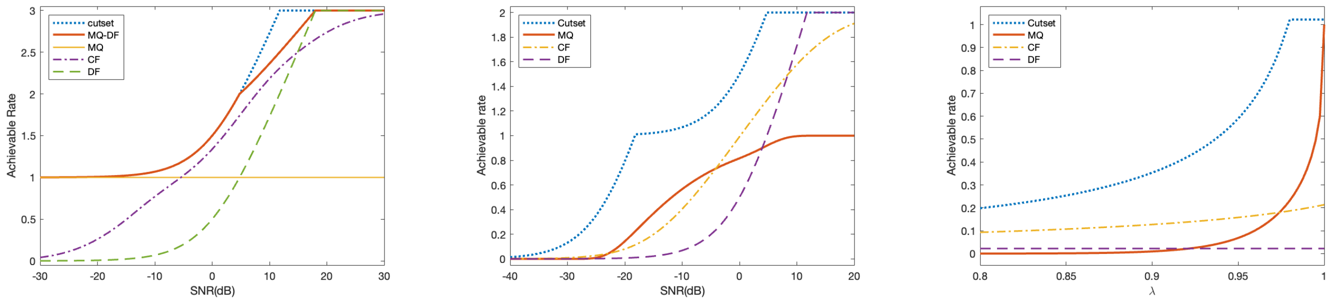
In the CF scheme, each relay compresses its observed sequence and forwards the resulting description to the receiver. The receiver recovers the transmitted message via joint decompression and decoding. Assuming Gaussian codebooks for encoding and Gaussian test channels for compression, the CF achievability bound can be expressed as

$$R_{\text{CF}}(R_1, R_2) = \max_{N_1, N_2 > 0} \min \left\{ \begin{matrix} \psi\left(\frac{(\alpha-\beta)^2 + \alpha^2 N_2 + \beta^2 N_1) P}{N_1 N_2 + N_1 + N_2}\right) \\ R_1 + \frac{1}{2}\log \frac{N_1(\beta^2 P + 1 + N_2)}{N_1 N_2 + N_1 + N_2} \\ R_2 + \frac{1}{2}\log \frac{N_2(\alpha^2 P + 1 + N_1)}{N_1 N_2 + N_1 + N_2} \\ R_1 + R_2 + \frac{1}{2}\log \frac{N_1 N_2}{N_1 N_2 + N_1 + N_2} \end{matrix} \right\}. \tag{18}$$

In Fig. 3(a) we plot these capacity bounds as functions of SNR for the case of $\alpha = -\beta = 1$, $R_1 = 2$, $R_2 = 1$. It is observed that MQ–DF outperforms all other schemes and matches the cut-set bound in both high-SNR and low-SNR regimes.

IV. Near-perfect Noise Correlation

In this section, we extend the MQ coding framework to the Gaussian primitive diamond channel with non-perfect correlation. The primary focus is on the regime of strong positive correlation, i.e., $\lambda \approx 1$. We demonstrate that a variant of MQ coding continues to outperform the benchmark DF and



(a) $\lambda = 1$, $\alpha = -\beta = 1$, $R_1 = 2$, $R_2 = 1$ (b) $\lambda = 0.99$, $\bar{R} = 1$, varying SNR (c) $\bar{R} = 1$, SNR $= -15$dB, varying λ.

Fig. 3. Comparison of capacity bounds for the Gaussian primitive diamond channel.

CF schemes. For simplicity, the scheme here has no DF component, and MQ is applied over the entire length-n block rather than to a single channel use. Specifically, MQ is used to induce a noisy point-to-point channel from the source to the destination, with error-correction applied to ensure reliability. In this section, we restrict attention to the setting where the relay-link capacities are the same and the relay channel gains have the same magnitude but opposite signs (the degenerate case $\alpha = \beta$ is excluded per discussion in Section III). In particular, we assume $R_1 = R_2 = \bar{R}$ and $\alpha = -\beta = 1$.

We first present a few capacity bounds for this channel. Since correlation does not affect the DF scheme, the DF achievability is identical to (17). The CF lower bound and the cut-set upper bound are functions of λ and can be derived analogously to the perfectly correlated case:

$$\text{Cutset}(\bar{R}; \lambda) = \min\left\{2\bar{R}, \bar{R} + \psi(P), \psi(\tfrac{2P}{1-\lambda})\right\}, \tag{19}$$

$$R_{\text{CF}}(\bar{R}; \lambda) = \max_{N>0} \min \left\{ \begin{array}{c} \psi(\frac{2P}{N+1-\lambda}) \\ \bar{R} + \frac{1}{2}\log\frac{N(P+1+N)}{(1+N)^2-\lambda^2} \\ 2\bar{R} + \frac{1}{2}\log\frac{N^2}{(1+N)^2-\lambda^2} \end{array} \right\}. \tag{20}$$

In the following, we construct a scalar MQ code for the example where $\bar{R} = 1$. When the correlation is near-perfect and the SNR is moderate, the MQ scheme is able to transmit one of two possible symbols in each channel use at a small probability of symbol error.

Fix input power $P > 0$ and let $\Delta = 4\sqrt{P}$. For a random message $M \in \{0, 1\}$, the input signal is $X = f_{\text{enc}}(M) = (2M - 1)\Delta/4$. It is easy to see that X takes value from $\{\pm\sqrt{P}\}$, hence the power constraint is satisfied. The relays forward the $(\Delta, 2)$-modulo quantization of a shifted version of their observation:

$$V_1 = \lceil Y_1 + \Delta \rfloor_{\Delta,2}, \quad V_2 = \lceil Y_2 + \Delta/2 \rfloor_{\Delta,2}.$$

The receiver declares $\hat{M} = (V_1 - V_2) \bmod 2$. An application of Lemma 1 together with some algebraic manipulations yield

$$\hat{M} = (M + \lceil \tilde{Z}_1 - \tfrac{M}{2}\Delta \rfloor_{\Delta,2} - \lceil \tilde{Z}_2 - \tfrac{M}{2}\Delta \rfloor_{\Delta,2}) \bmod 2,$$

where $\tilde{Z}_i = Z_i + \frac{3}{4}\Delta$ for $i \in \{1, 2\}$. Consequently, the symbol error probability of this scheme can be expressed as

$$P_e = \Pr\{\lceil \tilde{Z}_1 - \tfrac{M}{2}\Delta \rfloor_{\Delta,2} \neq \lceil \tilde{Z}_2 - \tfrac{M}{2}\Delta \rfloor_{\Delta,2})\},$$

where the probability on the right hand side is taken over $\tilde{Z}_1, \tilde{Z}_2$, and M. This probability can be computed via numerical integration. Define $\mathcal{R}_\Delta$ to be the following 2D-region consisting of countably many squares of side length Δ:

$$\mathcal{R}_\Delta = \bigcup_{i,j \in \mathbb{Z}} [i\Delta, (i+1)\Delta] \times [(i + 2j - 1)\Delta, (i + 2j)\Delta].$$

By definition, we see that $\mathcal{R}_\Delta = \{(x, y) : \lceil x \rfloor_{\Delta,2} \neq \lceil y \rfloor_{\Delta,2}\}$. Letting $\tilde{\boldsymbol{Z}} = (\tilde{Z}_1, \tilde{Z}_2)$, we further have

$$P_e = \frac{\Pr\{\tilde{\boldsymbol{Z}} \in \mathcal{R}_\Delta\} + \Pr\{\tilde{\boldsymbol{Z}} - \frac{\Delta}{2}\cdot\mathbf{1} \in \mathcal{R}_\Delta\}}{2} = \Pr\{\tilde{\boldsymbol{Z}} \in \mathcal{R}_\Delta\},$$

where $\mathbf{1}$ denotes the all-one vector in $\mathbb{R}^2$, and the last step makes use of the symmetry of $\tilde{\boldsymbol{Z}}$ and $\mathcal{R}_\Delta$.

The above procedure induces an effective binary symmetric channel from M to $\hat{M}$ with crossover probability P_e. Using an error-correcting code over this induced channel yields the achievable rate

$$R_{\text{MQ}}(1; \lambda) = I(M; \hat{M}) = 1 - h_b(P_e), \tag{21}$$

where $h_b(\cdot)$ is the binary entropy function.

The MQ achievability rate (21) and other known capacity bounds are compared in Fig. 3(b)-(c). We plot the bounds versus SNR for $\lambda = 0.99$ in Fig. 3(b). MQ outperforms both CF and DF over most of the low-SNR regime (SNR ≤ -5dB). In Fig. 3(c), the bounds are plotted versus λ at SNR $= -15$dB. In this case, the achievable rate of MQ deteriorates quickly when λ deviates from 1. Still, there is a regime where MQ surpasses both CF and DF, i.e., when $\lambda \geq 0.98$.

V. Conclusion

This paper demonstrates that MQ provides a powerful structured coding approach for the Gaussian primitive diamond channel with correlated noises. In the perfect correlation case, the proposed MQ and hybrid MQ–DF schemes exploit the channel determinism to yield improved achievability bounds. It allows an exact capacity characterization in certain parameter regimes. For the near-perfect noise correlation case, MQ can still outperform DF and CF strategies at low SNR.

References

[1] B. E. Schein, "Distributed coordination in network information theory," Ph.D. dissertation, Massachusetts Institute of Technology, 2001.

[2] A. Sanderovich, S. Shamai, Y. Steinberg, and G. Kramer, "Communication via decentralized processing," *IEEE Trans. Inf. Theory*, vol. 54, no. 7, pp. 3008–3023, July 2008.

[3] W. Kang and S. Ulukus, "Capacity of a class of diamond channels," *IEEE Trans. Inf. Theory*, vol. 57, no. 8, pp. 4955–4960, Aug. 2011.

[4] R. Tandon and S. Ulukus, "Diamond channel with partially separated relays," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, 2010, pp. 644–648.

[5] S. B. Shirin and G. Kramer, "Capacity bounds for a class of diamond networks," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, 2014, pp. 1196–1200.

[6] S. S. Bidokhti and G. Kramer, "Capacity bounds for diamond networks with an orthogonal broadcast channel," *IEEE Trans. Inf. Theory*, vol. 62, no. 12, pp. 7103–7122, Dec. 2016.

[7] H. Bagheri, A. S. Motahari, and A. K. Khandani, "On the capacity of the half-duplex diamond channel," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, 2010, pp. 649–653.

[8] W. Kang, N. Liu, and W. Chong, "The Gaussian multiple access diamond channel," *IEEE Trans. Inf. Theory*, vol. 61, no. 11, pp. 6049–6059, Nov. 2015.

[9] C. Huang, J. Jiang, and S. Cui, "On the achievable rates of the diamond relay channel with conferencing links," *IEEE Trans. Commun.*, vol. 60, no. 7, pp. 1838–1850, July 2012.

[10] W. Zhao, D. Y. Ding, and A. Khisti, "Capacity bounds for a class of diamond networks with conferencing relays," *IEEE Commun. Lett.*, vol. 19, no. 11, pp. 1881–1884, Nov. 2015.

[11] O. Aygün, E. Ozyilkan, and E. Erkip, "Neural compress-and-forward for the primitive diamond relay channel," *arXiv preprint arXiv:2512.07662*, 2025.

[12] A. Katz, M. Peleg, H. V. Poor, and S. Shamai, "The Gaussian primitive discrete time and filtered diamond channel: Correlated noise and dirty-paper coding," *IEEE Trans. Commun.*, Aug. 2025.

[13] ——, "Gaussian primitive diamond channel: Correlated noise at relays and relevant applications," in *Proc. IEEE Int. Conf. Microwaves, Commun., Antennas, Biomed. Eng. Electron. Syst. (COMCAS)*, 2024, pp. 1–4.

[14] Y.-H. Kim, "Capacity of a class of deterministic relay channels," *IEEE Trans. Inf. Theory*, vol. 54, no. 3, pp. 1328–1329, March 2008.

[15] R. K. Farsani and W. Yu, "Capacity bounds for broadcast channels with bidirectional conferencing decoders," *IEEE Trans. Inf. Theory*, vol. 71, no. 10, pp. 7484–7503, Oct. 2025.

[16] Y. Guo, S. C. Draper, and W. Yu, "Modulo quantization coding for Gaussian primitive relay channel with perfectly correlated noises," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, 2025.

[17] A. Lapidoth and Y. Yan, "The listsize capacity of the Gaussian channel with decoder assistance," *Entropy*, vol. 24, no. 1, p. 29, 2022.

[18] A. El Gamal and Y.-H. Kim, *Network Information Theory*. Cambridge University Press, 2011.