MIMO Sensing Beamforming Design with Low-Resolution Transceivers

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Abstract*—Adopting low-resolution hardware at transceivers in multi-input multi-output (MIMO) sensing systems can substantially reduce hardware costs and power consumption. This motivates us to study MIMO sensing systems with hardware constraints, specifically phase-only analog transmit antennas and low-resolution receive antennas. This paper adopts a Bayesian approach and aims to design low-complexity algorithms for the MIMO sensing beamforming problem while leveraging prior information about the target at each sensing stage. We formulate the problem of minimizing the Bayesian Cramér-Rao lower bound (BCRLB) for estimating a parameter of interest, and show that it has the structure of a weighted sum-of-ratios problem. For the case where the phase shifters at transmit antennas are continuous, we propose a novel linear transform that can transform a fractional function into a linear function. In this way, the original problem is turned into a sequence of sub-problems that can be solved in closed-form in each step with linear complexity in the number of antennas, making the iterative optimization process highly efficient. When the phase shifters are discrete, we propose a penalty-based convex-hull relaxation algorithm, which provides better performance than directly quantizing the solution of the continuous case, but at the cost of increased computational complexity. Numerical results demonstrate the effectiveness of the proposed algorithms.

I. Introduction

Multi-input multi-output (MIMO) sensing is a critical technique in integrated sensing and communications, which is a promising use case for future networks [1]. However, most of the current work in this area assumes the availability of high-resolution digital-to-analog converters (DACs) and analog-to-digital converters (ADCs), which are expensive from both implementation cost and power consumption perspectives. To address these issues, this paper investigates a MIMO sensing system with phase-only transmit antennas and low-resolution receivers, as shown in Fig. 1, and focuses on transmit beamformer design that accounts for the hardware constraints.

There are many prior research works investigating beamforming designs for MIMO sensing systems, where the Cramér-Rao lower bound (CRLB) [2], signal-to-noise ratio (SNR) [3], and beampattern [4] are typically used as sensing performance metrics. However, the issue of low-resolution hardware has not been fully accounted for in most earlier references. For example, in [5], the authors consider only the one-bit ADCs at receivers and use the SNR as a heuristic metric for sensing performance. In [6], the authors consider only the one-bit DACs at transmitters and adopt the CRLB as the sensing metric for a dual-function radar-communication system. In [7], the authors consider both one-bit DACs and ADCs in a MIMO

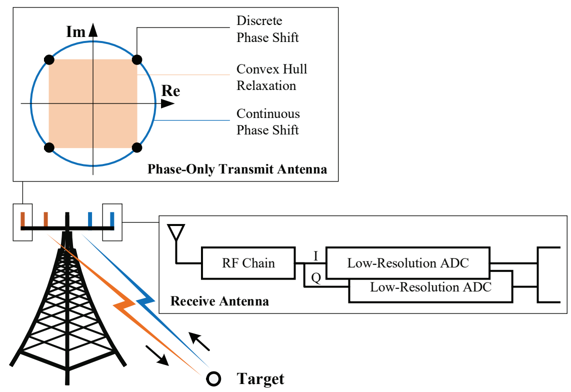

Fig. 1. A MIMO sensing system with low-resolution transceivers.

radar system, however, it focuses on target detection rather than parameter estimation. Differing from these prior works, this paper aims to address the parameter estimation problem by designing the MIMO sensing beamformers that account for low-resolution hardware constraints at both the transmitter and the receiver.

Moreover, none of the aforementioned references account for prior information in sensing. They do not properly address the issue that CRLB as a commonly adopted sensing metric depends on the exact values of the parameters to be estimated, which are not known. In practice, it is often possible to obtain and to track the prior distribution of the parameters of interest in a sequential manner so that the sensing objective can be successively refined. For this reason, this paper adopts the Bayesian CRLB (BCRLB) (e.g., [8]) as the sensing metric.

The main focus of this paper is to design MIMO sensing beamformers with low complexity, which is crucial for implementation in large-scale MIMO systems. Toward this end, this paper shows that the problem of minimizing the BCRLB for estimating a parameter of interest is a fractional programming problem. For the case where the phase shifters at the transmit antennas are continuous, we propose a novel linear transform that transforms a fractional function into a linear function. In this way, the original problem can be transformed into a sequence of sub-problems that can be solved in closed-form with complexity linear in the number of antennas, making the overall iterative optimization process highly efficient. When the phase shifters are discrete, we propose a penalty-based convex-hull relaxation algorithm, which has better performance than directly quantizing the solution of the continuous case, but at the cost of increased computational complexity.

These algorithms are highly effective for designing MIMO beamformers for radar sensing in a Bayesian setting.

II. System Model

Consider a MIMO sensing system equipped with N_T transmit antennas and N_R receive antennas, both arranged in a uniform linear array, as illustrated in Fig. 1. We assume that the receive array is equipped with low-resolution ADCs, e.g., one-bit, and the transmit array is phase-only. This paper considers both continuous and discrete phase shifters at the transmit antennas, for which the elements of transmit beamforming vector $\mathbf{x}$ are constrained as

$$|x_n| = 1,\ \ \forall n, \text{ if the phase shifter is continuous,} \tag{1}$$

$$x_n \in \mathcal{X},\ \forall n, \text{ if the phase shifter is discrete,} \tag{2}$$

where $\mathcal{X}$ is the discrete feasible set. Without loss of generality, the set $\mathcal{X}$ can be expressed as

$$\mathcal{X} = \{\exp\left(j\left[-\pi + (i-1)\Delta\right]\right) \mid i = 1, 2, \cdots, Q_T\}, \tag{3}$$

where $\Delta = 2\pi/Q_T$ if the transmit antenna provides Q_T phase shift levels. Both the constraints in (1) and (2) are non-convex.

This paper focuses on a sensing task of estimating the azimuth angle of a sensing target relative to the antenna array, denoted as η. The steering vectors from the antenna array to the target and from the target back to the antenna array can be expressed, respectively, as

$$\mathbf{h}(\eta) = \left[1, e^{j\tau\cos(\eta)}, \cdots, e^{j(N_T-1)\tau\cos(\eta)}\right]^{\mathsf{T}}, \tag{4}$$

$$\mathbf{v}(\eta) = \left[1, e^{j\tau\cos(\eta)}, \cdots, e^{j(N_R-1)\tau\cos(\eta)}\right]^{\mathsf{T}}, \tag{5}$$

where $\tau = 2\pi d/\omega$, ω represents the carrier wavelength, and d represents the spacing between the antennas (typically at half wavelength).

The received echo signal before digitization can then be expressed as

$$\mathbf{y} = \left(\alpha \mathbf{v}(\eta)\, \mathbf{h}^{\mathsf{T}}(\eta)\right)(\sqrt{p}\mathbf{x}) + \mathbf{n} \triangleq \sqrt{p}\mathbf{H}\mathbf{x} + \mathbf{n}, \tag{6}$$

where α represents the complex fading coefficient, p denotes the transmit power of each antenna, and $\mathbf{n}$ represents the noise with each element distributed as $\mathcal{CN}(0, \sigma_n^2)$. Here, we assume that the fading coefficient α is known for simplicity. In practice it can be estimated and tracked using various techniques, e.g., see [9] and references herein.

Assume that low-resolution ADCs are used at the receiver to quantize the received signal as follows:

$$\mathbf{r} = \mathcal{Q}(\mathbf{y}) = \mathcal{Q}(\sqrt{p}\mathbf{H}\mathbf{x} + \mathbf{n}), \tag{7}$$

where $\mathcal{Q}(\cdot)$ is the Q_R-bit quantization operation. Assuming the use of automatic gain control at the receiver, we can model the quantization process using an additive quantization noise model (AQNM) [10]–[13] as follows:

$$\mathbf{r} = \kappa\mathbf{y} + \mathbf{n}_{\mathrm{q}} = \kappa\sqrt{p}\mathbf{H}\mathbf{x} + \kappa\mathbf{n} + \mathbf{n}_{\mathrm{q}}, \tag{8}$$

where κ is the quantization gain, defined as $\kappa = 1-\nu$, and ν is the normalized mean squared quantization error. The relation between Q_R and ν is given in [10]–[13]. Here, $\mathbf{n}_{\mathrm{q}}$ denotes the additive Gaussian quantization noise that is uncorrelated with $\mathbf{y}$, and the covariance matrix of $\mathbf{n}_{\mathrm{q}}$ is given by

$$\begin{aligned}\mathbf{R}_{\mathrm{q}} &= \kappa(1-\kappa)\operatorname{diag}[\mathbf{R}_{\mathbf{y}}] \\ &= \kappa(1-\kappa)\operatorname{diag}\left[p(\mathbf{H}\mathbf{x})(\mathbf{H}\mathbf{x})^{\mathsf{H}} + \sigma_n^2\mathbf{I}\right].\end{aligned} \tag{9}$$

Note that in this model, the quantization noise power is proportional to the power of the signal to be quantized. This is due to the use of automatic gain control that scales the dynamic range of the quantizer, while the number of quantization levels remains fixed. From (9), one can observe that the quantization noise variance is also dependent on the transmit beamforming vector $\mathbf{x}$, increasing the difficulty of the beamformer design.

III. Sensing Scheme and Performance Metric

We adopt an active sensing scheme. Specifically, the MIMO sensing system forms the transmit beamforming vector according to the prior distribution of the target's azimuth angle, and then updates its posterior distribution based on the received echo signals. The posterior distribution after each sensing stage is used as the prior distribution in the next sensing stage to design the subsequent beamformer. This process is repeated over multiple stages.

Conventionally, CRLB has been widely used to characterize the estimation performance of deterministic parameters. It has also been widely adopted as an alternative metric to the MSE when the MSE is difficult to compute. But the computation of CRLB depends on the exact values of parameters. In practice, however, only a prior distribution of the parameters is known. Therefore, we cannot directly employ the classic CRLB as the optimization objective. Instead, this paper uses the BCRLB as the metric for estimating η, as given by

$$\mathsf{BCRLB}(\eta) = \frac{1}{\mathbb{E}_{q(\eta)}[\mathsf{FI}(\eta)] + \mathsf{FIP}(\eta)}, \tag{10}$$

where $q(\eta)$ denotes the prior distribution of η, $\mathsf{FI}(\eta)$ denotes the Fisher information of η, and $\mathsf{FIP}(\eta)$ represents the Fisher information from prior information and is independent on $\mathbf{x}$. The Fisher information $\mathsf{FI}(\eta)$ is given by [14]

$$\mathsf{FI}(\eta) = 2\left[\left(\kappa\sqrt{p}\dot{\mathbf{H}}\mathbf{x}\right)^{\mathsf{H}}\mathbf{\Sigma}^{-1}\left(\kappa\sqrt{p}\dot{\mathbf{H}}\mathbf{x}\right)\right], \tag{11}$$

where $\dot{\mathbf{H}}$ represents the derivative of $\mathbf{H}$ with respect to η, and $\mathbf{\Sigma}$ denotes the covariance matrix of the quantization noise plus the additive white noise, expressed as

$$\mathbf{\Sigma} = \kappa(1-\kappa)\operatorname{diag}\left[p(\mathbf{H}\mathbf{x})(\mathbf{H}\mathbf{x})^{\mathsf{H}} + \sigma_n^2\mathbf{I}\right] + \kappa^2\sigma_n^2\mathbf{I}. \tag{12}$$

One can observe from (10) that minimizing $\mathsf{BCRLB}(\eta)$ over $\mathbf{x}$ is equivalent to maximizing $\mathbb{E}_{q(\eta)}[\mathsf{FI}(\eta)]$. Thus, we define the following metric for estimation performance:

$$\mathbb{E}_{q(\eta)}[\mathcal{S}(\eta)] \triangleq \mathbb{E}_{q(\eta)}\left[\frac{\mathsf{FI}(\eta)}{2p\kappa^2}\right]. \tag{13}$$

Then, the problem of transmit beamforming design is turned into a maximization of $\mathbb{E}_{q(\eta)}[\mathcal{S}(\eta)]$. In the sequel, we formulate the problems for both the cases of continuous and discrete phase shifters, and design the transmit beamforming vector in each case.

IV. Beamforming Design for MIMO Sensing

Based on the performance metric established in the previous section, we now formulate the beamformer design problem as

$$\texttt{(PO)}: \underset{\mathbf{x}}{\text{maximize}} \quad \mathbb{E}_{q(\eta)}[\mathcal{S}(\eta)] \tag{14a}$$
$$\text{subject to} \quad |x_n| = 1 \text{ or } x_n \in \mathcal{X}, \ \forall n. \tag{14b}$$

The problem (PO) has three main numerical difficulties:

- The optimization variable $\mathbf{x}$ is embedded inside an expectation operation.
- The objective function contains multi-dimensional ratios.
- The feasible set is non-convex.

To tackle the first difficulty, we adopt uniform sampling for approximating the expectation and rewrite problem (PO) as

$$\underset{\mathbf{x}}{\text{maximize}} \quad \sum_{i=1}^{L} \frac{q_i}{\sum_{j=1}^{L} q_j} \mathcal{S}(\eta_i) \tag{15a}$$
$$\text{subject to} \quad |x_n| = 1 \text{ or } x_n \in \mathcal{X}, \ \forall n, \tag{15b}$$

where L denotes the number of samples, i represents the index of the i-th sample, and $q_i \triangleq q(\eta_i)$ represents the probability density of the prior.

In the next two subsections, we address the above optimization problem for the scenarios of continuous and discrete phase shifts, respectively. Specifically, we propose algorithms to deal with the issue that the objective function contains weighted sum of ratios.

A. Beamforming Design for Continuous Case

In this subsection, we focus on the continuous-phase-shift case. The corresponding problem is formulated as

$$\texttt{(P1)}: \underset{\mathbf{x}}{\text{maximize}} \quad \sum_{i=1}^{L} q_i \left[\left(\dot{\mathbf{H}}_i \mathbf{x}\right)^{\mathsf{H}} \boldsymbol{\Sigma}_i^{-1} \left(\dot{\mathbf{H}}_i \mathbf{x}\right) \right] \tag{16a}$$
$$\text{subject to} \quad |x_n| = 1, \ \forall n. \tag{16b}$$

To solve problem (P1) in an efficient manner, we propose a *constant-modulus linear transform* in the following theorem.

Theorem 1: Consider a weighted sum-of-ratios maximization problem with constant-modulus constraints as

$$\underset{\mathbf{x}}{\text{maximize}} \quad \sum_{i} w_i g_i(\mathbf{x}) \tag{17a}$$
$$\text{subject to} \quad |x_n| = 1, \ \forall n, \tag{17b}$$

where $g_i(\mathbf{x})$ is a multi-dimensional ratio defined as

$$g_i(\mathbf{x}) \triangleq (\mathbf{A}_i \mathbf{x})^{\mathsf{H}} \mathbf{R}_i^{-1} (\mathbf{A}_i \mathbf{x}), \tag{18}$$

and the denominator matrix $\mathbf{R}_i$ is defined as

$$\mathbf{R}_i \triangleq \operatorname{diag}\left[\sum_{k \in \mathcal{R}_i} (\mathbf{A}_k \mathbf{x}) (\mathbf{A}_k \mathbf{x})^{\mathsf{H}} \right] + \mathbf{C}_i. \tag{19}$$

Here, the matrix $\mathbf{C}_i$ is positive definite so that the denominator matrix $\mathbf{R}_i$ is invertible.

Then, the problem (17) is equivalent to

$$\underset{\mathbf{x},\mathbf{z},\boldsymbol{\lambda}}{\text{maximize}} \quad \sum_{i} w_i f_i(\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda}_i) \tag{20a}$$
$$\text{subject to} \quad |x_n| = 1, \ |z_n| = 1, \ \forall n \tag{20b}$$

where the transformed objective function is given by

$$f_i(\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda}_i) \triangleq 2\,\mathfrak{Re}\left(\mathbf{x}^{\mathsf{H}}\left[(\delta_i \mathbf{I} - \mathbf{D}_i)\mathbf{z} + \mathbf{A}_i^{\mathsf{H}} \boldsymbol{\lambda}_i\right]\right) + c_i(\mathbf{z}, \boldsymbol{\lambda}_i), \tag{21}$$

$\mathbf{z}$ and $\boldsymbol{\lambda}_i$ are auxiliary variables introduced to decouple the numerator and denominator of $g_i(\mathbf{x})$, the matrix $\mathbf{D}_i$ is

$$\mathbf{D}_i = \sum_{k \in \mathcal{R}_i} \mathbf{A}_k^{\mathsf{H}} \operatorname{diag}\left[\boldsymbol{\lambda}_i \boldsymbol{\lambda}_i^{\mathsf{H}}\right] \mathbf{A}_k, \tag{22}$$

δ_i represents the trace of the matrix $\mathbf{D}_i$, and

$$c_i(\mathbf{z}, \boldsymbol{\lambda}_i) = \mathbf{z}^{\mathsf{H}} \mathbf{D}_i \mathbf{z} - 2\delta_i N_T - \boldsymbol{\lambda}_i^{\mathsf{H}} \mathbf{C}_i \boldsymbol{\lambda}_i. \tag{23}$$

For fixed $\mathbf{x}$, the optimal auxiliary variables for maximizing the objective in (20) are given as

$$\mathbf{z}^\star = \mathbf{x}, \tag{24}$$
$$\boldsymbol{\lambda}_i^\star = \mathbf{R}_i^{-1} \mathbf{A}_i \mathbf{x}. \tag{25}$$

Proof: The proof is given in Appendix A. ■

This theorem uses a key technique in [15] that turns a quadratic optimization over $\mathbf{x}$ with unit-modulus constraints into a linear optimization. This is made possible by taking advantage of the unit-modulus property of $\mathbf{x}$. Please also note that this theorem can be generalized by removing the diagonal extraction operations $\operatorname{diag}(\cdot)$ in (19) and (22).

According to Theorem 1, problem (P1) can be equivalently transformed into the following problem:

$$\texttt{(P2)}: \underset{\mathbf{x},\mathbf{z},\boldsymbol{\lambda}}{\text{maximize}} \quad \sum_{i=1}^{L} q_i f_i(\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda}) \tag{26a}$$
$$\text{subject to} \quad |x_n| = 1, \ |z_n| = 1, \ \forall n. \tag{26b}$$

The transformed objective function is given by

$$f_i(\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda}) = 2\,\mathfrak{Re}\left(\mathbf{x}^{\mathsf{H}}\left[(\delta_i \mathbf{I} - \mathbf{M}_i)\mathbf{z} + \dot{\mathbf{H}}_i^{\mathsf{H}} \boldsymbol{\lambda}_i\right]\right) + c_i(\mathbf{z}, \boldsymbol{\lambda}_i), \tag{27}$$

where the matrix $\mathbf{M}_i$ is given by

$$\mathbf{M}_i = p\left(\kappa - \kappa^2\right) \mathbf{H}_i^{\mathsf{H}} \operatorname{diag}\left[\boldsymbol{\lambda}_i \boldsymbol{\lambda}_i^{\mathsf{H}}\right] \mathbf{H}_i, \tag{28}$$

and δ_i is the trace of $\mathbf{M}_i$. Note that $c_i(\mathbf{z}, \boldsymbol{\lambda}_i)$ has no influence on the updating of $\mathbf{x}$, thus is omitted here. A key advantage of the problem reformulation (P2) is that the objective function is now linear with respect to the optimization variable. This gives arise to efficient algorithm for solving problem (P2) with per-iteration complexity which is *linear* in the number of antennas.

Now, problem (P2) can be solved in an iterative manner. More specifically, when $\mathbf{x}$ is held fixed, the auxiliary variables $\mathbf{z}$ and $\boldsymbol{\lambda}_i$ can be updated by

$$\mathbf{z}^\star = \mathbf{x}, \tag{29}$$
$$\boldsymbol{\lambda}_i^\star = \boldsymbol{\Sigma}_i^{-1} \dot{\mathbf{H}}_i \mathbf{x}. \tag{30}$$

When $\mathbf{z}$ and $\boldsymbol{\lambda}_i$ are held fixed, $\mathbf{x}$ can be updated by solving the following linear programming problem:

$$\underset{\mathbf{x}}{\text{maximize}} \quad \sum_{i=1}^{L} q_i f_i(\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda}) \tag{31a}$$

$$\text{subject to} \quad |x_n| = 1, \ \forall n. \tag{31b}$$

It is easy to demonstrate that the optimal solution of the above subproblem is given by

$$\mathbf{x}^\star = \exp\left(j \arg\left[\sum_{i=1}^{L} q_i \left((\delta_i \mathbf{I} - \mathbf{M}_i) \mathbf{z} + \dot{\mathbf{H}}_i^{\mathsf{H}} \boldsymbol{\lambda}_i \right) \right] \right). \tag{32}$$

Since all the variables have closed-form updating solutions, the proposed algorithm is highly efficient.

B. Beamforming Design for Discrete Case

Now, consider the case of discrete phase shifts, the problem can be formulated as

$$\textbf{(P3)}: \underset{\mathbf{x}}{\text{maximize}} \quad \sum_{i=1}^{L} q_i \left[\left(\dot{\mathbf{H}}_i \mathbf{x} \right)^{\mathsf{H}} \boldsymbol{\Sigma}_i^{-1} \left(\dot{\mathbf{H}}_i \mathbf{x} \right) \right] \tag{33a}$$

$$\text{subject to} \quad x_n \in \mathcal{X}, \ \forall n. \tag{33b}$$

One direct approach to solving problem (P3) is to quantize the solution of problem (P1) obtained in the previous subsection. However, this can cause a performance loss. In this subsection, to obtain a better performance and allow the obtained solution to directly satisfy the discrete constraints, we propose an algorithm based on a penalty method with convex-hull relaxation [16]. By adding a penalty term into the objective function and relaxing the discrete feasible set to its convex-hull, the problem (P3) is transformed into the following problem:

$$\underset{\mathbf{x},\mathbf{z}}{\text{maximize}} \quad \sum_{i=1}^{L} q_i \left[\left(\dot{\mathbf{H}}_i \mathbf{x} \right)^{\mathsf{H}} \boldsymbol{\Sigma}_i^{-1} \left(\dot{\mathbf{H}}_i \mathbf{x} \right) \right] - \mu \|\mathbf{x} - \mathbf{z}\|^2 \tag{34a}$$

$$\text{subject to} \quad x_n \in \mathsf{conv}(\mathcal{X}), \ |z_n| = 1, \ \forall n, \tag{34b}$$

where $\mu > 0$ determines the extent of penalty, and the penalty term $\|\mathbf{x} - \mathbf{z}\|^2$ equals to zero if and only if $x_n \in \mathcal{X}$ and $\mathbf{z} = \mathbf{x}$. Thus, by tuning the penalty coefficient, the penalty can force the optimized solution to be the vertices of the convex hull. Next, we use the quadratic transform technique [17] to solve the problem (34). The first step is to transform (34) into

$$\textbf{(P4)}: \underset{\mathbf{x},\mathbf{z},\boldsymbol{\lambda}}{\text{maximize}} \quad \sum_{i=1}^{L} q_i g_i(\mathbf{x}, \boldsymbol{\lambda}) - \mu \|\mathbf{x} - \mathbf{z}\|^2 \tag{35a}$$

$$\text{subject to} \quad x_n \in \mathsf{conv}(\mathcal{X}), \ |z_n| = 1, \ \forall n, \tag{35b}$$

where the transformed objective function is given by

$$g_i(\mathbf{x}, \boldsymbol{\lambda}) = 2\,\mathfrak{Re}\left(\boldsymbol{\lambda}_i^{\mathsf{H}} \dot{\mathbf{H}}_i \mathbf{x} \right) - \boldsymbol{\lambda}_i^{\mathsf{H}} \boldsymbol{\Sigma}_i \boldsymbol{\lambda}_i. \tag{36}$$

Now, problem (P4) can be solved in an iterative manner. More specifically, when $\mathbf{x}$ is held fixed, the auxiliary variables $\mathbf{z}$ and $\boldsymbol{\lambda}_i$ can be updated by

$$\mathbf{z}^\star = \exp\left[j \arg(\mathbf{x}) \right], \tag{37}$$

$$\boldsymbol{\lambda}_i^\star = \boldsymbol{\Sigma}_i^{-1} \dot{\mathbf{H}}_i \mathbf{x}. \tag{38}$$

When $\mathbf{z}$ and $\boldsymbol{\lambda}_i$ are held fixed, the subproblem of updating $\mathbf{x}$ is a convex quadratic programming problem as follows:

$$\underset{\mathbf{x}}{\text{maximize}} \quad \sum_{i=1}^{L} q_i g_i(\mathbf{x}, \boldsymbol{\lambda}) - \mu \|\mathbf{x} - \mathbf{z}\|^2 \tag{39a}$$

$$\text{subject to} \quad x_n \in \mathsf{conv}(\mathcal{X}), \ \forall n. \tag{39b}$$

The optimal solution can be easily obtained using a standard optimization solver.

C. Complexity Analysis

In this subsection, we briefly analyze the complexity of the proposed method and compare it with that of a benchmark of projected gradient ascent method which is widely adopted and can guarantee the convergence on a unit-sphere [18]. The updating rule for the optimization variable $\mathbf{x}$ is given by

$$\mathbf{x}^{t+1} = \mathsf{Proj}\left[\mathbf{x}^t + \mathsf{step} \times \nabla_{\mathbf{x}} \mathsf{Obj}(\mathbf{x}) \right]. \tag{40}$$

As to the discrete case, we directly quantize the continuous solution obtained by (40) to the discrete feasible set.

The computational complexity of the linear transform-based algorithm is $\mathcal{O}(N_T)$ in each iteration since all the variables have closed-form solutions. Thus, the linear transform-based algorithm for the continuous-phase problem is highly efficient.

As for the projected gradient ascent method, the complexity is also $\mathcal{O}(N_T)$ per iteration. But it is nontrivial to choose appropriate step sizes in the projected gradient algorithm, which can affect the converge speed and performance. As seen in the next section, the projected gradient method with common heuristic for choosing the step size takes much longer to converge as compared to the linear transform-based method.

Regarding the convex-hull relaxation-based algorithm, the complexity in each iteration is at least $\mathcal{O}(N_T^2)$ since we need to solve a convex quadratic programming problem over a convex polyhedron in each iteration. Thus, this algorithm for the discrete-phase problem has a higher complexity.

V. Numerical Results

In this section, we provide numerical results to illustrate the effectiveness of our proposed algorithms. The simulation environment is set as follows (unless otherwise specified).

- The ADCs are one-bit with the normalized mean squared quantization error $\nu = 0.3634$.
- The discrete phase shift level $Q_T = 4$ and $\mathcal{X}$ is given by

$$\mathcal{X} = \frac{1}{\sqrt{2}} \left\{ 1 + j, 1 - j, -1 + j, -1 - j \right\}. \tag{41}$$

- The transmit power is set such that $pN_T/\sigma_n^2 = 10$ dB.
- The numbers of transmit and receive antennas are set as $N_T = N_R = 16$.

In addition, we consider three scenarios as follows:

1) The azimuth angle of the point target has a uniform prior distribution in the range $[40^\circ, 80^\circ]$.
2) The azimuth angle has a Gaussian prior distribution with mean 60° and standard deviation 10°.
3) The azimuth angle is almost deterministic at 60°.

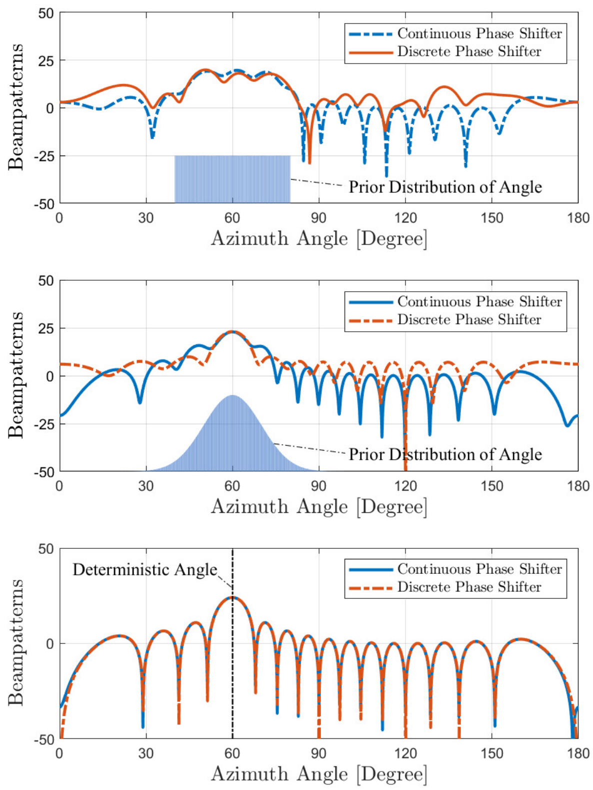

Fig. 2. Transmit beampatterns with different prior distributions of η.

TABLE I
RUNTIME AND ITERATION NUMBERS OF DIFFERENT ALGORITHMS.

	LT	PG	PCH
Runtime for 100 Iterations [s]	0.4014	3.0346	33.0872
Number of Iterations to Converge	60	145	15

* LT: Linear Transform, PG: Projected Gradient Ascent, PCH: Penalty Convex Hull.

Before illustrating the sensing performance, we first show the efficiency of the proposed algorithms. Table I gives the runtime and the number of iterations to converge for the different algorithms in the first considered scenario. We observe that the proposed linear transform method demonstrates a significantly faster runtime than the other two algorithms, while the penalty and convex hull relaxation based method is the slowest, which aligns with the complexity analysis presented in the previous section.

Next, we show the optimized transmit beampattern, which is defined as

$$\mathcal{Q}\left(\eta\right)=\left|\mathbf{h}^{\mathsf{T}}(\eta)\,\mathbf{x}\right|^{2}, \tag{42}$$

in Fig. 2. The designed transmit beampatterns are interpretable. From Fig. 2, it can be observed that the solutions of the optimization problem produce beams that are aligned with the prior distributions of the target angle.

To further illustrate the sensing performance of the proposed algorithm, we show the evolution of the posterior distributions after several iterations of sensing stages in Fig. 3. The posterior probability function of the $(t+1)$-th sensing stage is computed as follows:

$$\begin{aligned}\mathsf{Pr}\left(\eta\,|\,\mathbf{r}_{t+1}\right) &\propto \mathcal{L}\left(\mathbf{r}_{t+1}\,|\,\eta\right)\cdot\mathsf{Pr}\left(\eta\,|\,\mathbf{r}_{t}\right)\\ &=\mathcal{CN}\left(\mathbf{r}_{t+1}\,|\,\kappa\sqrt{p}\mathbf{H}\mathbf{x},\boldsymbol{\Sigma}\right)\cdot\mathsf{Pr}\left(\eta\,|\,\mathbf{r}_{t}\right),\end{aligned} \tag{43}$$

where $\mathcal{L}\left(\mathbf{r}_{t+1}\,|\,\eta\right)$ denotes the likelihood function of η given the received signal $\mathbf{r}_{t+1}$, and $\mathsf{Pr}\left(\eta\,|\,\mathbf{r}_{t}\right)$ represents the posterior distribution from the previous iteration, which is used as the prior distribution $q\left(\eta\right)$ for the current iteration. We show the result of the first scenario under consideration in Fig. 3. The actual angle of the target is at 70°. We plot the posterior distribution of η after three iterations using the transmit beamformer designed by the proposed algorithm. We can see from Fig. 3 that as the number of sensing stages increases, the posterior distribution of η rapidly converges to a highly concentrated distribution with a peak at the true sensing angle. This shows that the active sensing scheme is highly effective.

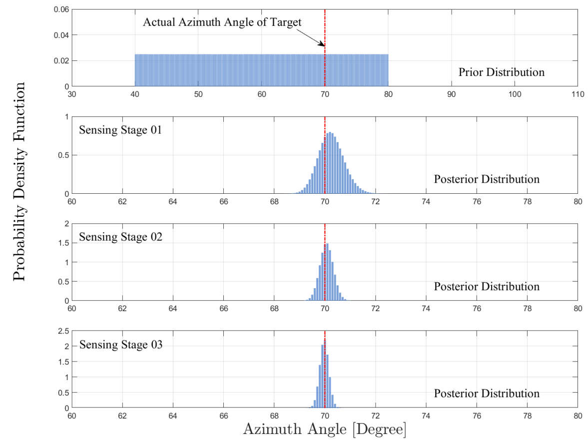

Fig. 3. Posterior distributions of three iterations with the transmit beamforming vectors designed by the proposed algorithm.

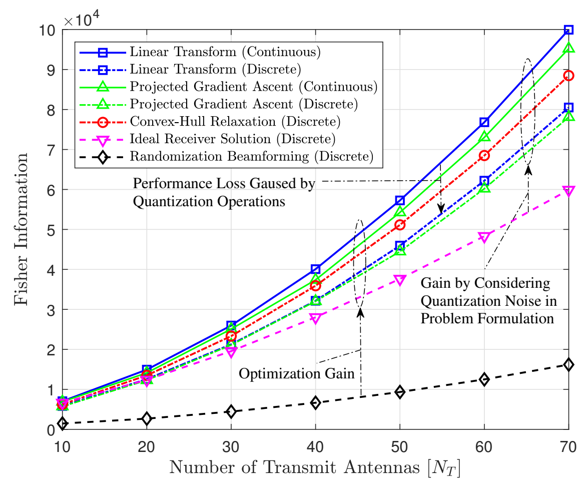

Fig. 4. Fisher information versus the number of transmit antennas N_T using different optimization algorithms.

In Fig. 4, we show the results of different algorithms versus the number of transmit antennas in the range of $[10, 70]$, in the first scenario under consideration. One can observe that the proposed algorithms perform better than the benchmark and the result of randomization beamforming. Most importantly, the results show that better performance can be achieved by considering the quantization noise in the problem formulation. In other words, if we overlook the influence of quantization at receivers and treat the one-bit receiver as an ideal receiver in the design of the beamformer, it would lead to a noticeable performance loss. In addition, it can be observed from Fig. 4 that as the number of antennas increases, the performance loss caused by quantization operations also increases. Hence, when the phase shifters are discrete, one can make informed choices between the linear transform based method and the penalty and convex hull relaxation based method according to performance and complexity tradeoff.

VI. Conclusion

This paper proposes methodologies for transmit beamforming designs for a MIMO sensing system with low-resolution transceivers. The problem is formulated as minimizing the BCRLB for estimating the target's azimuth angle. This problem is a fractional programming problem. We first propose a novel linear transform technique to tackle the case of continuous phase shifts. Then, we propose a penalty and convex hull relaxation based algorithm to tackle the case of discrete phase shifts. These algorithms are highly effective for designing MIMO transmit beamformers for the sensing problem that accounts for hardware constraints.

Appendix A
Proof of Theorem 1

According to the quadratic transform in [17], a lower bound of (18) can be established as follows:

$$g_i(\mathbf{x}) \geq 2\,\mathfrak{Re}\Big[(\mathbf{A}_i\mathbf{x})^{\mathsf{H}}\boldsymbol{\lambda}_i\Big] - \boldsymbol{\lambda}_i^{\mathsf{H}}\mathbf{R}_i\boldsymbol{\lambda}_i. \tag{44}$$

The equality in (44) is achieved at

$$\boldsymbol{\lambda}_i = \mathbf{R}_i^{-1}\mathbf{A}_i\mathbf{x}. \tag{45}$$

Then, based on the following equation,

$$\mathbf{a}^{\mathsf{H}}\operatorname{diag}\big[\mathbf{b}\mathbf{b}^{\mathsf{H}}\big]\,\mathbf{a} = \mathbf{b}^{\mathsf{H}}\operatorname{diag}\big[\mathbf{a}\mathbf{a}^{\mathsf{H}}\big]\,\mathbf{b}, \tag{46}$$

we have

$$\begin{aligned}\boldsymbol{\lambda}_i^{\mathsf{H}}\mathbf{R}_i\boldsymbol{\lambda}_i &= \mathbf{x}^{\mathsf{H}}\left(\sum_{k\in\mathcal{R}_i}\mathbf{A}_k^{\mathsf{H}}\operatorname{diag}\Big[\boldsymbol{\lambda}_i\boldsymbol{\lambda}_i^{\mathsf{H}}\Big]\mathbf{A}_k\right)\mathbf{x} + \boldsymbol{\lambda}_i^{\mathsf{H}}\mathbf{C}_i\boldsymbol{\lambda}_i \\ &\triangleq \mathbf{x}^{\mathsf{H}}\mathbf{D}_i\mathbf{x} + \boldsymbol{\lambda}_i^{\mathsf{H}}\mathbf{C}_i\boldsymbol{\lambda}_i.\end{aligned} \tag{47}$$

We eliminate the quadratic term in (47) by making use of the fact that for unit-modulus variables $\mathbf{x}$ and $\mathbf{z}$,

$$\mathbf{x}^{\mathsf{H}}(\delta_i\mathbf{I})\,\mathbf{x} = \mathbf{z}^{\mathsf{H}}(\delta_i\mathbf{I})\,\mathbf{z} = \delta_i N_T. \tag{48}$$

Specifically, we apply [19, Eq. (26)] to (47), which is repeated here that

$$\begin{aligned}\mathbf{x}^{\mathsf{H}}\mathbf{D}_i\mathbf{x} \leq\ & \mathbf{x}^{\mathsf{H}}\mathbf{L}_i\mathbf{x} + \mathbf{z}^{\mathsf{H}}(\mathbf{L}_i - \mathbf{D}_i)\,\mathbf{z} \\ &+ 2\,\mathfrak{Re}\big(\mathbf{x}^{\mathsf{H}}(\mathbf{D}_i - \mathbf{L}_i)\,\mathbf{z}\big),\end{aligned} \tag{49}$$

where $\mathbf{L}_i \succeq \mathbf{D}_i$, and the equality is achieved at $\mathbf{z} = \mathbf{x}$. Then, following [15], we replace $\mathbf{L}_i$ with $\delta_i\mathbf{I}$, where δ_i is the trace of $\mathbf{D}_i$ so that $\delta_i\mathbf{I} \succeq \mathbf{D}_i$. By combining with (48), we obtain

$$\begin{aligned}g_i(\mathbf{x}) &\geq f_i(\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda}_i) \\ &\triangleq 2\,\mathfrak{Re}\left(\mathbf{x}^{\mathsf{H}}\left[(\delta_i\mathbf{I} - \mathbf{D}_i)\,\mathbf{z} + \mathbf{A}_i^{\mathsf{H}}\boldsymbol{\lambda}_i\right]\right) + c_i(\mathbf{z}, \boldsymbol{\lambda}_i),\end{aligned} \tag{50}$$

where $c_i(\mathbf{z}, \boldsymbol{\lambda}_i)$ is given in (23). The equality in the above is achieved when $\mathbf{z} = \mathbf{x}$ and $\boldsymbol{\lambda}_i = \mathbf{R}_i^{-1}\mathbf{A}_i\mathbf{x}$. This shows the equivalence of (17) and (20) under unit modulus constraints.

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