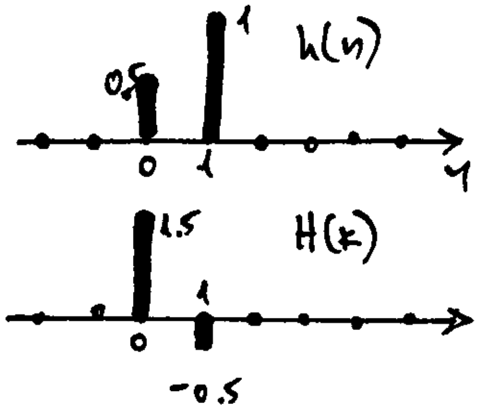
COURSE ELE 4315

(Instructor: D. Hatzinakos)

Solutions (Problem set on DFT/FFT)

1). a)



2-DFT : $H(k)=\sum_{n=0}^{1} h(n)e^{-j\frac{2\pi k n}{2}}$ $\quad k=0,1.$

$H(0)=\sum_{n=0}^{1} h(n) = 0.5+1 = \underline{\underline{1.5}}$

$H(1)=\sum_{n=0}^{1} h(n)e^{-j\frac{2\pi k}{2}} = h(0)+h(1)\cdot e^{-j\pi} =$

$= 0.5 + 1(-1) = \underline{\underline{-0.5}}$

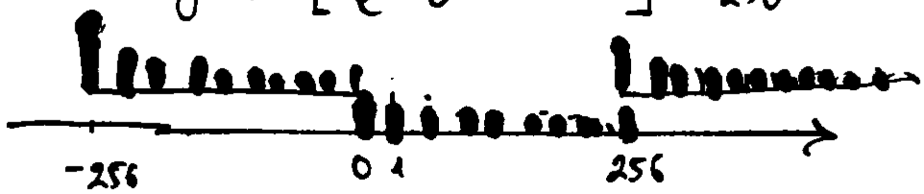
b) $\tilde{y}(n) = x(n) * h(n)$

$y(n) = x(n) \underset{256}{\circledast} h(n)$

$\left.\begin{array}{l} x(n),\ n=0,\dots,255 \text{ is of length } 256 \\ h(n),\ n=0,1 \text{ is of length } 2 \end{array}\right\} \Rightarrow$ $\tilde{y}(n)$ is of length $256+2-1=257$ (from $n=0,\dots,256$)

$y(n)$ is of length 256 ($n=0,\dots,255$

Also, $y(n) = \left[\sum_{\ell} \tilde{y}(n+\ell N)\right] R_{256}(n)$



(overlapping over one point)

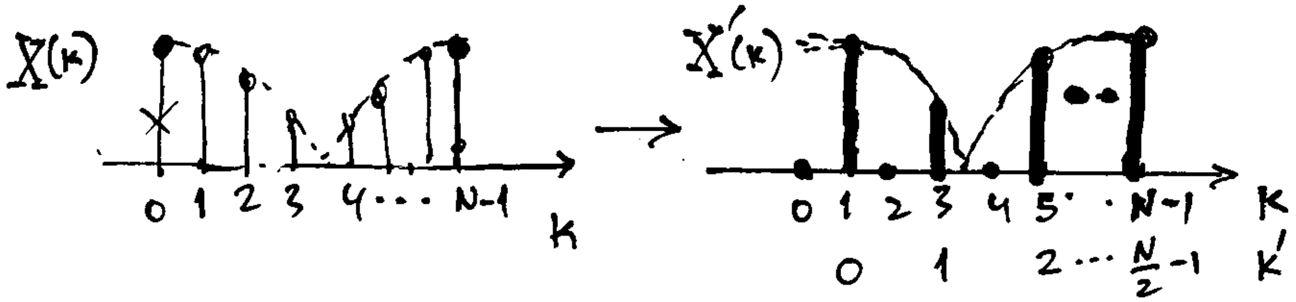
Thus $y(n) \equiv \tilde{y}(n)$ for $n=1,\dots,255$

c) Since we take 256-FFT and $256 = 2^8$

Decimation in time FFT – 8 stages

" " Frequency FFT – 8 "

(2) $x(n)$, $n = 0, 1, \ldots, N = 2^\nu$ (real sequence)



So $X'(k) = X(2k'+1)$, $k' = 0, 1, \ldots, N/2 - 1$

$$\Rightarrow X'(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n(2k'+1)}{N}} = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{4\pi n k'}{N}} e^{-j\frac{2\pi n}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{2k'n} W_N^{n} = \sum_{n=0}^{N-1} x(n)\, W_{N/2}^{k'n} W_N^{n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_{N/2}^{k'n} W_N^{n} + \sum_{n=N/2}^{N-1} x\left(n+\frac{N}{2}\right) W_{N/2}^{k'\left(n+\frac{N}{2}\right)} W_N^{\left(n+\frac{N}{2}\right)}$$

From properties: $W_{N/2}^{k'n}$, $-W_N^{n}$

$$\Rightarrow X'(k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^{n}\, W_{N/2}^{k'n}.$$

$k = 0, 1, \ldots, \frac{N}{2} - 1$.

Thus: (1) Form sequence $\left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^{n}$ for $n = 0, \ldots, \frac{N}{2} - 1$

(2) Estimate $N/2$ DFT sequence.

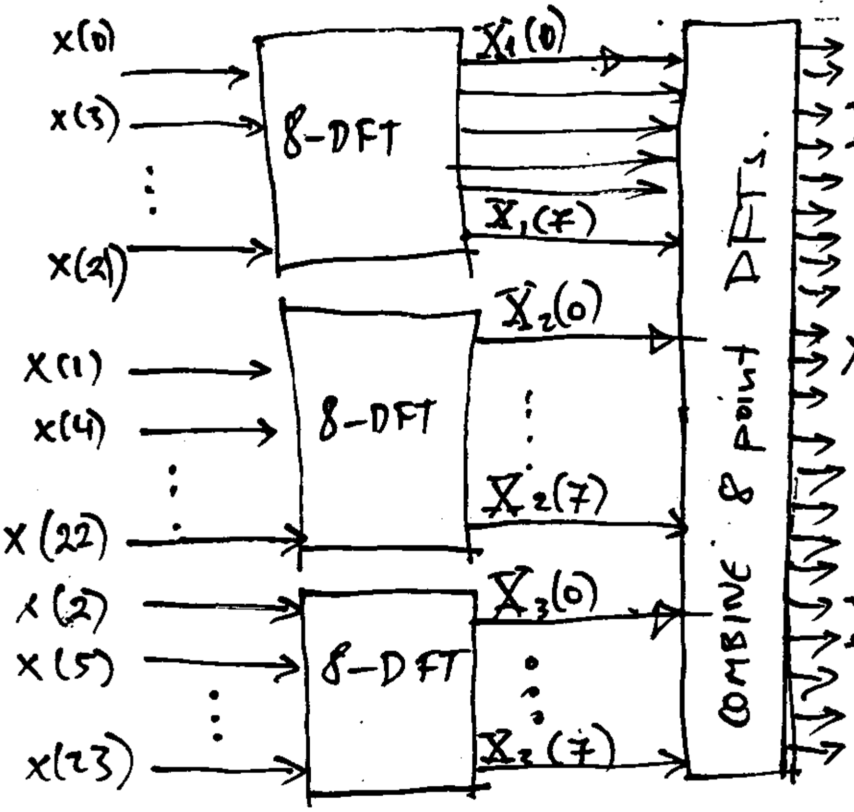
3) We have 3 8-FFT chips.
Compute a 24 point DFT $k=0,\dots,23$

$$X(k)=\sum_{n=0}^{23} x(n)W_{24}^{nk}=\sum_{n=3m} x(n)\,W_{24}^{nk}+\sum_{n=3m+1} x(n)W_{24}^{nk}+\sum_{n=3m+2} x(n)\,W_{24}^{nk}$$

$$=\sum_{m=0}^{7} x(3m)\,W_{24}^{3mk}+\sum_{m=0}^{7} x(3m+1)\,W_{24}^{(3m+1)k}+\sum_{m=0}^{7} x(3m+2)W_{24}^{(3m+2)k}$$

$$=\sum_{m=0}^{7} x(3m)\,W_{8}^{mk}+\sum_{m=0}^{7} x(3m+1)\,W_{8}^{mk}\cdot W_{24}^{k}+\sum_{m=0}^{7} x(3m+2)W_{8}^{mk}\,W_{24}^{2k}$$

$$=\underbrace{\sum_{m=0}^{7} x(3m)\,W_{8}^{mk}}_{\substack{8\text{-DFT}\\ X_1(k)}}+W_{24}^{k}\underbrace{\sum_{m=0}^{7} x(3m+1)W_{8}^{mk}}_{\substack{8\text{-DFT}\\ X_2(k)}}+W_{24}^{2k}\underbrace{\sum_{m=0}^{7} x(3m+2)W_{8}^{mk}}_{\substack{8\text{-DFT}\\ X_3(k)}}$$



$X(k)=X_1(k)+W_{24}^{k}X_2(k)+W_{24}^{2k}X_3(k)$

$k=0,1,\dots,7$

$X(k+8)=X_1(k)+W_{24}^{k+8}X_2(k)+W_{24}^{(2k+16)}X_3(k)$

$k=0,\dots,7$

$X(k+16)=X_1(k)+W_{24}^{k+16}X_2(k)+W_{24}^{(2k+32)}X_3(k)$

$k=0,\dots,7.$

DECIMATION IN TIME

(4)

$x(n)$ in bit reversal	Outputs of first stage	Outputs of second stage	Outputs of third stage $X(k)$ in normal order
$\frac{1}{2}$	$X_1(0) = 1/2$	$X_2(0) = 1$	$X_3(0) = 2$
0	$X_1(1) = 1/2$	$X_2(1) = \frac{1}{2}(1+W_8^2)$	$X_3(1) = \frac{1}{2}(1+W_8^1+W_8^2+W_8^3)$
$\frac{1}{2}$	$X_1(2) = 1/2$	$X_2(2) = 0$	$X_3(2) = 0$
0	$= 1/2$	$X_2(3) = \frac{1}{2}(1-W_8^2)$	$X_3(3) = \frac{1}{2}(1-W_8^2+W_8^3-W_8^5)$
1/2	$= 1/2$	$X_2(4) = 1$	$X_3(4) = 0$
0	$= 1/2$	$X_2(5) = \frac{1}{2}(1+W_8^2)$	$X_3(5) = \frac{1}{2}(1-W_8^1+W_8^2-W_8^3)$
1/2	$= 1/2$	$X_2(6) = 0$	$X_3(6) = 0$
0	$X_1(7) = 1/2$	$X_2(7) = \frac{1}{2}(1-W_8^2)$	$X_3(7) = \frac{1}{2}(1-W_8^2-W_8^3+W_8^5)$

DECIMATION IN FREQUENCY

$x(n)$ in normal order			$X(k)$ (in bit reversal)
1/2	$\frac{1}{2}$	1	2
1/2	$\frac{1}{2}$	1	0
1/2	$\frac{1}{2}$	0	0
1/2	$\frac{1}{2}$	0	0
0	$\frac{1}{2}$	$\frac{1}{2}(1+W_8^2)$	$\frac{1}{2}(1+W_8^1+W_8^2+W_8^3)$
0	$\frac{1}{2}W_8^1$	$\frac{1}{2}(W_8^1+W_8^3)$	$\frac{1}{2}(1-W_8^1+W_8^2-W_8^3)$
0	$\frac{1}{2}W_8^2$	$\frac{1}{2}(1-W_8^2)$	$\frac{1}{2}(1-W_8^2+W_8^3-W_8^5)$
0	$\frac{1}{2}W_8^3$	$\frac{1}{2}(W_8^3-W_8^5)$	$\frac{1}{2}(1-W_8^2-W_8^3+W_8^5)$