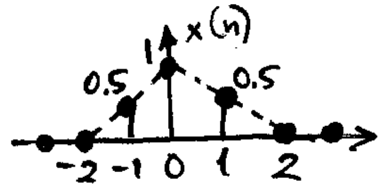
Review: Let $x(n) \xrightarrow{DTFT} X(\omega)$, the N-DFT maps one period of $\tilde{x} = \sum_{\ell} x(n+\ell N)$ to any period of $X\left(\omega = \frac{2\pi k}{N}\right)$

LECTURE 9

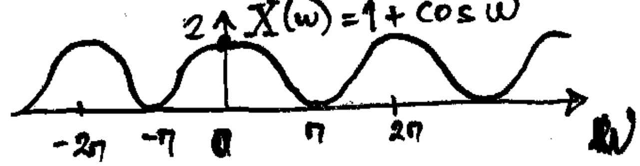
ZERO-PADDING / DIGITAL INTERPOLATION (UP SAMPLING)

Ex

Let

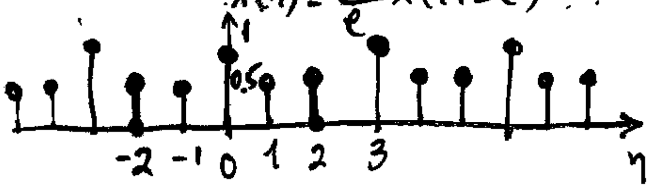


$\xrightarrow{DTFT}$

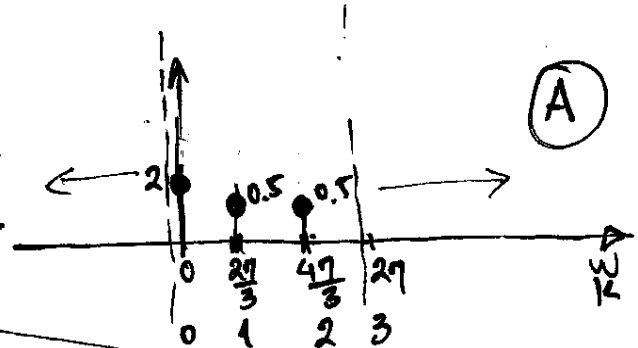


A) Obtain 3-point DFT of $x(n)$

$\tilde{x}(n) = \sum_{\ell} x(n+3\ell)$



3-DFT

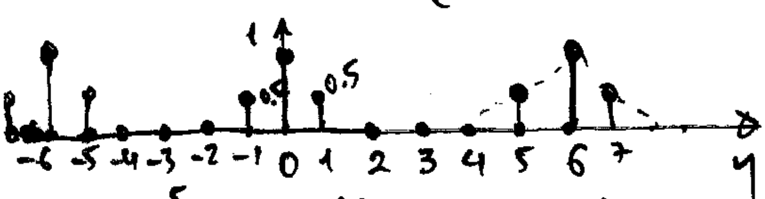


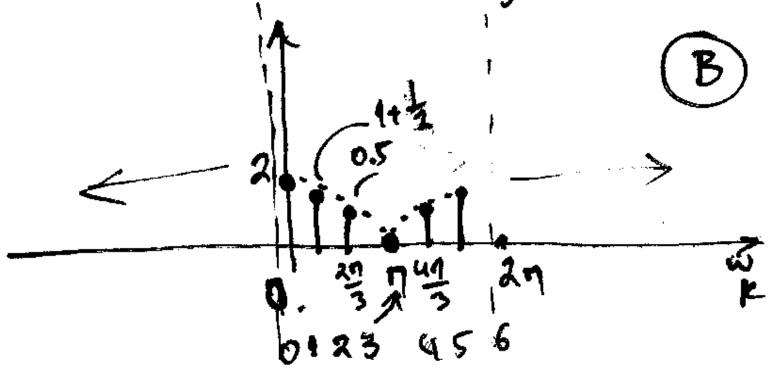
$$\tilde{X}(k) = \sum_{n=0}^{2} \tilde{x}(n) e^{-j\frac{2\pi k n}{3}} = \sum_{n=0}^{2} \tilde{x}(n) e^{-j\frac{2\pi k n}{3}} =$$

$$= 1 \cdot e^{j0} + 0.5 e^{-j\frac{2\pi k}{3}} + 0.5 e^{-j4\pi k/3} = 1 + 0.5\left(e^{-j\frac{2\pi k}{3}} + e^{j\frac{2\pi k}{3}}\right) = 1 + \cos\left(\frac{2\pi k}{3}\right) \quad k = 0, 1, 2$$

B) Obtain 6-point DFT of $x(n)$

$\tilde{x}(n) = \sum_{\ell} x(n+6\ell)$





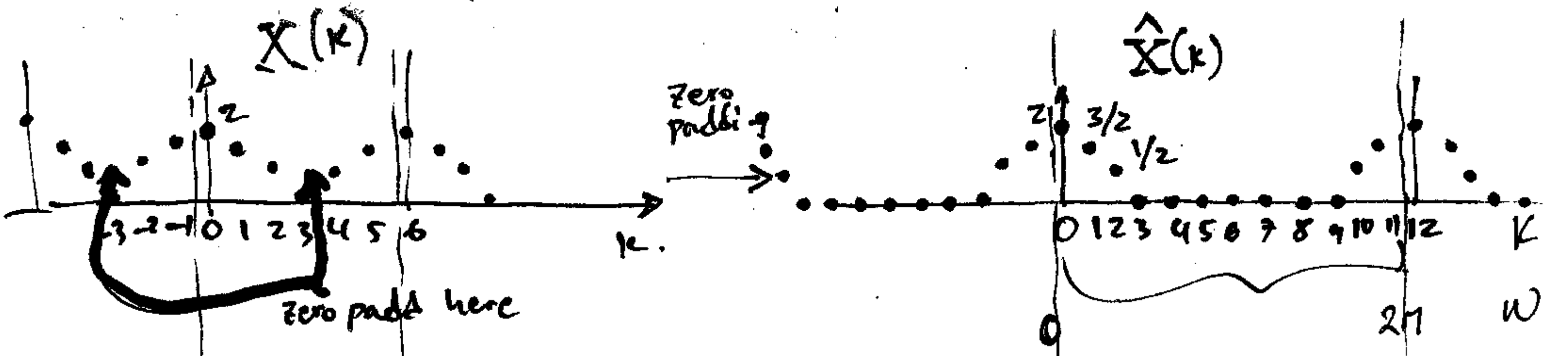
$$\tilde{X}(k) = \sum_{n=0}^{5} \tilde{x}(n) e^{-j\frac{2\pi k n}{6}} = 1 + 0.5 e^{-j\frac{\pi k}{3}} + 0 + 0 + 0 + 0.5 e^{-j\frac{5\pi k}{3}} = 1 + 0.5\left[e^{-j\frac{\pi k}{3}} + e^{-j\frac{5\pi k}{3}}\right]$$

$$= 1 + 0.5\left[e^{-j\frac{\pi k}{3}} + e^{j\frac{\pi k}{3}}\right] = 1 + \cos\left(\frac{\pi k}{3}\right), \quad k = 0, 1, 2, 3, 4, 5$$

* By zero-padding $\tilde{x}(n) = \sum_{\ell} x(n+3\ell)$ with 3-zeros/period, we have interpolated 3 values in $3\text{-}\tilde{X}(k) \to 6\text{-}\tilde{X}(k)$

In general by zero padding $\tilde{x}(n) = \sum_{\ell} x(n+N\ell)$ with L zeros ($L = K \cdot N$)
Then, we have interpolated values in $N\text{-}\tilde{X}(k)$. (K new values between any two values of N-DFT)
(Up sampling in frequency domain)

Now let us zero-padd the 6-DFT with 6 zeros and take 12-IDFT,



12-IDFT

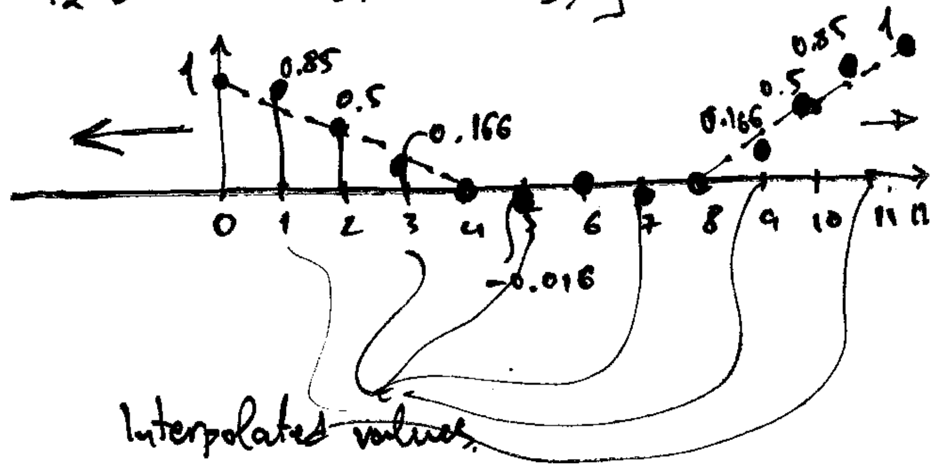
$$\hat{x}(n)=\frac{1}{12}\sum_{k=0}^{11}\hat{X}(k)e^{j\frac{2\pi kn}{12}}=\frac{1}{12}\left[2+\frac{3}{2}\left(e^{j\frac{\pi n}{6}}+e^{-j\frac{\pi n}{6}}\right)+0.5\left(e^{j\frac{2\pi n}{6}}+e^{-j\frac{2\pi n}{6}}\right)\right]=$$

$$=\frac{1}{12}\left[2+3\cos\left(\frac{\pi n}{6}\right)+\cos\left(\frac{\pi n}{3}\right)\right]$$

$n=0,1,\ldots,11$

$\rightarrow \hat{x}(0)=\frac{1}{2}$
$\hat{x}(1)\simeq 0.425$
$\hat{x}(2)\simeq 0.25$
$\hat{x}(3)=0.083$
$\hat{x}(4)=0.$
$\hat{x}(5)=-0.0081$
$\hat{x}(6)=0$
$\hat{x}(7)\doteq -0.0081$
$\hat{x}(8)=0$
$\hat{x}(9)=0.083$
$\hat{x}(10)=0.25$
$\hat{x}(11)=0.425$

$\times\ 2 =$



* By zero padding the N-DFT of $x(n)$ with $L=KN$ zeros outside the range $-\frac{N}{2}+1,\ldots,\frac{N}{2}$ (main spectrum range) and then taking $(K+1)N$-IDFT and multiplying by $K+1$ $\rightarrow$ We obtain K interpolated values between any two values of $x(n)$ (~~UP-SAMPLING~~ in time-domain)

In general: Interpolation (L values in $x(n)$)

$$\underset{n=0,1,\ldots,N-1}{X(n)}\rightarrow\boxed{N\text{-DFT}}\xrightarrow{X(k)}\boxed{\text{zero padd } L \text{ zeros}}\xrightarrow{\hat{X}(k)}\boxed{(N+L)\text{-IDFT}}\rightarrow\boxed{\text{scale by } \frac{N+L}{N}}\xrightarrow{\hat{x}(n)}$$

If $L=K\cdot N$, K values are interp betw any two val. of $x(n)$

zero-padding $x(n)$ $\xrightarrow{\text{DFT}}$ Interpolation (up sampling) $X(k)$

Interpolation (up sampling) $x(n)$ $\xleftarrow{\text{IDFT}}$ zero padding $X(k)$

Note: zero-padding $X(k)$ with ∞ zeros is equivalent to infinite sampling $(T\to 0)$ $\rightarrow$ $x_a(t)$ reconstruction from $x(n)$