# LECTURE 7 ELE 431S

## The DTFT (discrete-time Fourier Transform, aperiodic discrete signals)

Let $x_a(t)$ aperiodic $\rightarrow$ $X_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\,\delta(t-nT_s)$ aperiodic)

Then:

$$X_s(\Omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} X_a(nT_s)\,\delta(t-nT_s)\,e^{-j\Omega t}\,dt = \sum_{n=-\infty}^{\infty} X_a(nT_s) \int_{-\infty}^{\infty} \delta(t-nT_s)\,e^{-j\Omega t}\,dt =$$

$$= \sum_{n=-\infty}^{\infty} X_a(nT_s)\,e^{-j\Omega nT_s} \quad \underline{\text{sampling Th.}} \quad \frac{1}{T_s}\sum_{\ell=-\infty}^{\infty} X_a\left(\Omega - \ell\frac{2\pi}{T_s}\right)$$

or $X(\omega) = \sum_{n} x(n)\,e^{-j\omega n}$ $\qquad \frac{1}{T}\sum_{\ell} X_a\left(\frac{\omega - \ell 2\pi}{T}\right)$

Since $X_s(\Omega)$ : periodic $\left(\frac{2\pi}{T_s}\right)$, $\Omega$ continuous from Fourier Series

$$x(n) = X_a(nT_s) = \frac{1}{2\pi/T}\int_{-\pi/T}^{\pi/T} X_s(\Omega)\,e^{j\Omega nT_s}\,d\Omega \overset{\omega=\Omega T}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\,e^{j\omega n}\,d\omega$$

So:

DTFT pair:

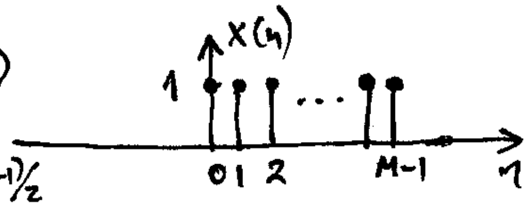
$$\boxed{\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x(n)\,e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\,e^{j\omega n}\,d\omega \end{aligned}}$$

$x(n)$: aperiodic discrete

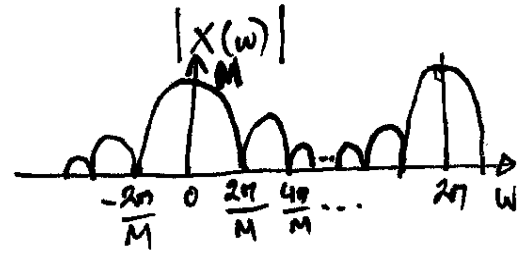
$X(\omega)$: continuous periodic $(2\pi)$

Note: $x(n)$ can be written as $x(n) = \sum_{k} x(k)\,\delta(n-k)$ $\quad$ ($T=1$, assumption)

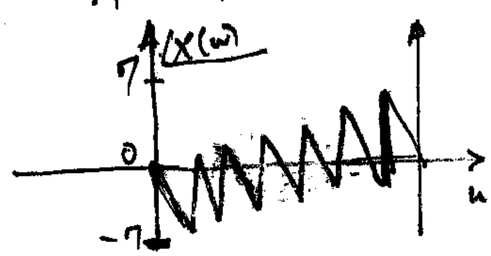
**Ex** Let $x(n) = u(n) - u(n-M) = \sum_{k=0}^{M-1} \delta(n-k)$



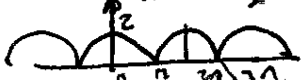

then, $X(\omega) = \sum_{n=0}^{M-1} e^{-j\omega n} = \frac{1-e^{-j\omega M}}{1-e^{-j\omega}} = \frac{\sin(\omega M/2)}{\sin(\omega/2)}\,e^{-j\omega(M-1)/2}$

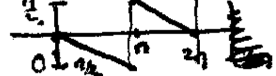



$$|X(\omega)| = \left|\frac{\sin(\omega M/2)}{\sin(\omega/2)}\right|, \quad \angle X(\omega) = -\omega\frac{(M-1)}{2} + \text{or}\begin{cases} 0 \\ \pi \end{cases}$$




$M=2$ $\quad X(\omega) = \frac{\sin \omega}{\sin\frac{\omega}{2}} e^{-j\frac{\omega}{2}} = 2\cos\frac{\omega}{2}\cdot e^{-j\frac{\omega}{2}} = 2\left|\cos\frac{\omega}{2}\right| e^{-j\frac{\omega}{2}}$ $\quad \angle X(\omega) = -\frac{\omega}{2} + \begin{cases}0\\ \pi\end{cases}$



# Section 8.1

$$\tilde{x}_a(t)=\tilde{x}_a(t+kT_0)$$
$$\tilde{x}_a(nT_s)=\tilde{x}_a(nT_s+kT_0)=\tilde{x}_a[(n+N)T_s] \Rightarrow$$

## THE DTFS (or DFS) Discrete-time Fourier Series

Let $\tilde{x}_a(t)$ be periodic $(T_0)$: Then if $kT_0=NT_s$, $\tilde{x}_s(t)$ also periodic (per. $NT_s$)

Then from Fourier Series

$$\tilde{X}_s\left(\Omega=\frac{2\pi k}{NT_s}\right)=\frac{1}{NT_s}\int_0^{NT_s}\sum_{n=-\infty}^{\infty}\tilde{x}_a(nT_s)\delta(t-nT_s)e^{-j\frac{2\pi kt}{NT_s}}dt=\frac{1}{NT}\sum_{n=-\infty}^{\infty}\tilde{x}_a(nT_s)\int_0^{NT_s}\delta(t-nT_s)e^{-j\frac{2\pi kt}{NT_s}}dt=$$

$$=\frac{1}{NT_s}\sum_{n=0}^{N-1}\tilde{x}_a(nT_s)e^{-j\frac{2\pi kn}{N}}$$ periodic with period **N** in k

$$\text{or } \tilde{X}_s(\Omega)=\sum_{k=-\infty}^{\infty}\tilde{X}_s\left(\frac{2\pi k}{NT_s}\right)\delta\left(\Omega-\frac{2\pi k}{NT_s}\right)\ \underline{\underline{\text{sampling Theorem}}}\ \frac{1}{T}\sum_{\ell=-\infty}^{\infty}X_a\left(\Omega-\ell\frac{2\pi}{T_s}\right)$$

$\sim$ denotes periodicity

Since $X_s(\Omega)$ periodic $\left(\frac{2\pi}{T_s}\right)$ Fourier Series $\longrightarrow$

$$\tilde{x}(n)=\tilde{x}_a(nT_s)=\frac{1}{2\pi/T_s}\int_0^{2\pi/T_s}\sum_{k=-\infty}^{\infty}\tilde{X}_s\left(\frac{2\pi k}{NT_s}\right)\delta\left(\Omega-\frac{2\pi k}{NT_s}\right)e^{j\Omega nT_s}d\Omega=\cdots=T\sum^{N-1}\tilde{X}_s\left(\frac{2\pi k}{NT_s}\right)e^{j\frac{2\pi kn}{N}}$$
$$=\frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1}x(n)e^{-j\frac{2\pi kn}{N}}\right]e^{j\frac{2\pi kn}{N}}$$

[Also by placing $\frac{1}{N}x(n)=y(n)\Rightarrow y(n)\cdots$

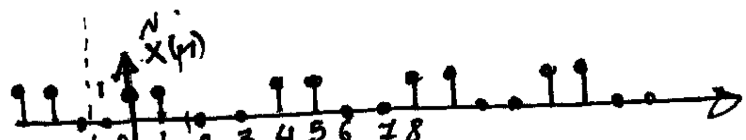
So in normalized freq. $(\omega=\Omega T)$ (or placing $T=1$)

DTFS. pair

$$\boxed{\begin{aligned}\tilde{X}(k)&=\sum_{n=0}^{N-1}\tilde{x}(n)e^{-j\frac{2\pi kn}{N}}\\ \tilde{x}(n)&=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}(k)e^{j\frac{2\pi kn}{N}}\\ k,n&=0,\dots N-1 \quad \text{or over } \langle N\rangle\end{aligned}}$$

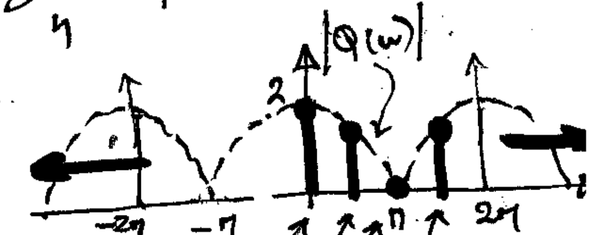
$\left.\begin{matrix}\tilde{x}(n)\\ \tilde{X}(k)\end{matrix}\right\}$ Discrete and Periodic with period N.

**Ex.**




period $N=4$.

Let


$$Q(\omega)=\sum_{n=0}^{1}e^{-j\omega n}=\frac{\sin\omega}{\sin\omega/2}e^{-j\omega/2}=2\cos\frac{\omega}{2}e^{-j\omega/2}$$

$$\tilde{X}(k)=Q\left(\omega=\frac{2\pi k}{N}=\frac{\pi k}{2}\right)$$

$X(k) \leftarrow k=0,1,2,3$