Lecture 6

Examples

(a) Sampling and reconstruction of a sinusoidal signal.

Let $x_a(t) = \cos(6\pi t) = \cos(2\pi 3 t) = \frac{1}{2} e^{j2\pi 3t} + \frac{1}{2} e^{-j2\pi 3t}$

Let $T_s = 0.1$ sec ($F_s = 10$ Hz)

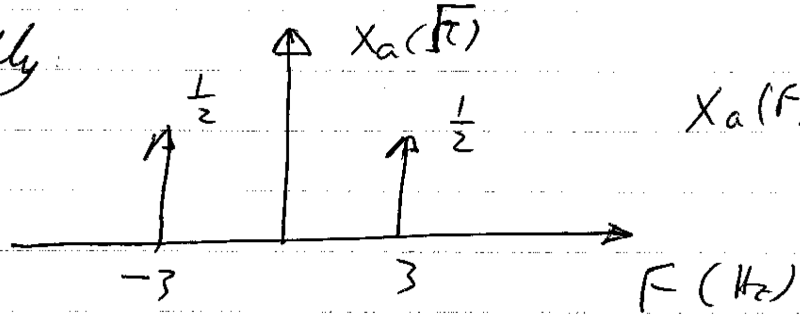
Then $x[n] = x_a(nT_s) = \cos(0.6\pi n) = \cos(2\pi 0.3 n) = \frac{1}{2} e^{j0.6\pi n} + \frac{1}{2} e^{-j0.6\pi n}$

$x_a(t)$ is periodic in time with period $T = \frac{1}{3}$ sec

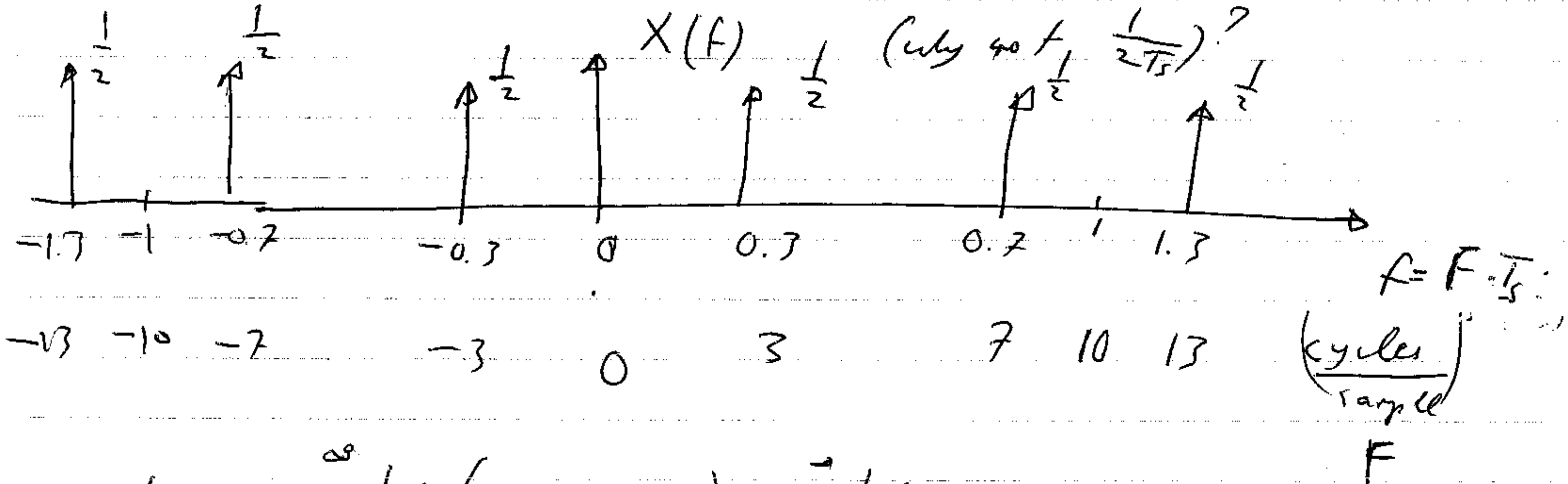
$x[n]$ is periodic in frequency with period equal to 1

also since $f = \frac{F}{F_s} = \frac{3}{10} = 0.3$ $x[n]$ is also periodic in n with period 10 samples.

Graphically



$X_a(F) = \frac{1}{2}\delta(F-3) + \frac{1}{2}\delta(F+3)$



$$X(f) = \sum_{k=-\infty}^{\infty} \frac{1}{2}\delta(f-0.3-k) + \sum_{k=-\infty}^{\infty} \frac{1}{2}(f+0.3-k)$$

Now let $x_a(t) = \cos(14\pi t)$ and let again $T_s = 0.1$ sec
In this case $x_a(t) = \cos(2\pi 7 t)$ and $F_s = 10$ Hz

Then, $x[n] = x_a(nT_s) = \cos(1.4\pi n) = \cos(2\pi\, 0.7 n) =$

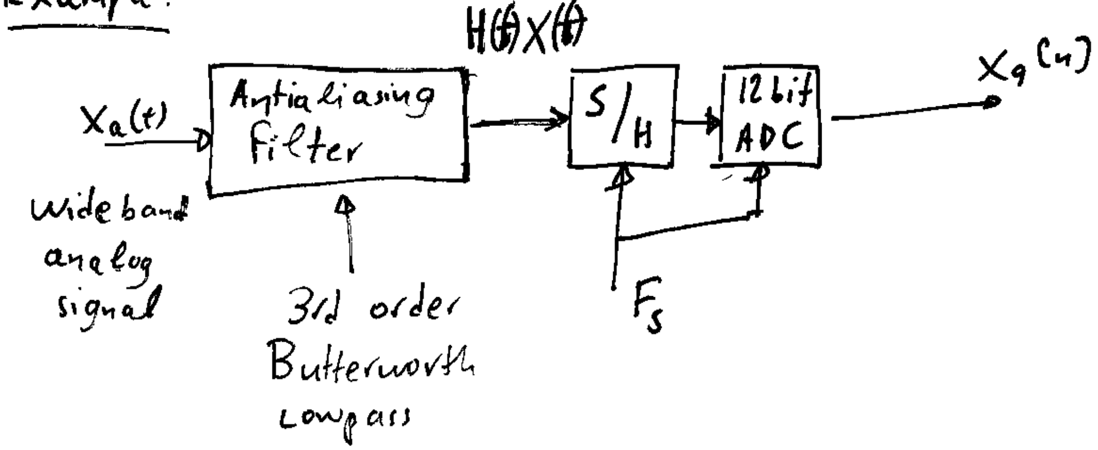
$$= \cos(2\pi(1-0.3)n) = \cos(2\pi n - 2\pi\, 0.3 n)$$
$$= \cos(-2\pi\, 0.3 n) = \cos(2\pi\, 0.3 n) = \cos(0.6 n)$$

So in this case aliasing is present. Due to aliasing the sequence $x_a(t)$ will be reconstructed as

$\cos(6\pi t)$ instead of $\cos(14\pi t)$

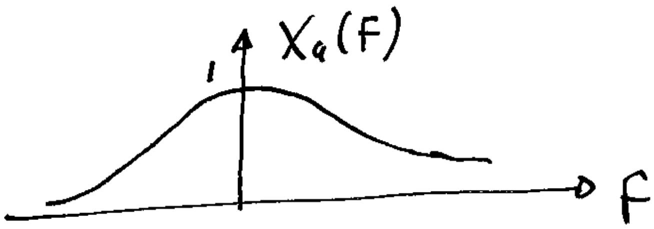
In other words with sampling frequency $F_s = 10$ Hz the two sequences $\cos(0.6\pi n)$ and $\cos(1.4\pi n)$ are not distinguishable.

Example.



① Assuming that the band of interest extends from 0 to 4 KHz estimate the sampling Frequency F_s so that the aliasing error in the passband is negligible compared to the quantization error level.

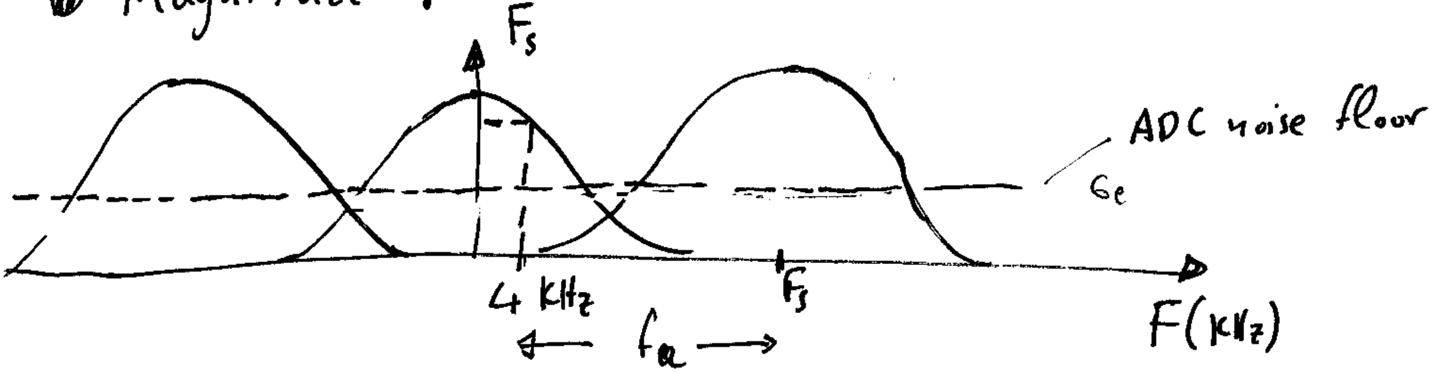
SOLUTION :



Butterworth filter with $F_c = 4$ KHz

$$H(F) = \frac{1}{\left[1 + \left(\frac{F}{4 \cdot 10^3}\right)^6\right]^{\frac{1}{2}}}$$

② Magnitude Spectrum of sampled signal



→ Assuming that the signal fills the dynamic range of the quantizer V_{fs}, then $\Delta = \frac{V_{fs}}{2^{12}}$

→ At 4 kHz the aliasing error is

$$F_s \cdot \frac{1}{\left[1+\left(\frac{F_s-4}{4}\right)^6\right]^{\frac{1}{2}}} \cdot \left|X_a(F_s-4)\right| \quad \text{(A)}$$

The quantization noise RMS level $= \sqrt{\frac{\Delta^2}{12}} = \frac{V_{fs}}{\sqrt{12}\cdot 2^B} = \frac{V_{fs}}{\sqrt{12}\, 2^{12}} = 6e$ (B)

So we want (A) < (B)

numerical example: Let the input signal be a long truncated sinusoid, then $V_{fs} = 2A$, $|X(F_s-4)| = \frac{A}{\sqrt{2}}$

Thus,

$$\frac{F_s \; A/\sqrt{2}}{\left(1+\left(\frac{F_s-4}{4}\right)^6\right)^{\frac{1}{2}}} < \frac{2A}{\sqrt{12}\, 2^{12}} \Rightarrow 1+\frac{(F_s-4)^6}{4^6} > \sqrt{2}\cdot 12 \cdot 2^{24} F_s \Rightarrow$$

$$\Rightarrow (F_s-4)^6 > \sqrt{2}\cdot 12\; 2^{36} \cdot F_s - 2^{12}$$

$$\Rightarrow F_s \sim \left(\sqrt{2}\; 12 \cdot 2^{36}\right)^{\frac{1}{5}} \simeq 2^8 \text{ kHz} \simeq 256 \text{ kHz}$$