Lect. 31) **ELE 431 S (1991-1992)** (D. Hatzinakos)

IIR FILTER DESIGN BY IMPULSE INVARIANCE METHOD

Basic idea: Preserve the impulse response in D.T. (discrete time) domain. That is choose $\{h(n)\}$ such that:

$$\boxed{h(n) = h_a(nT)} \qquad \begin{array}{l} n = 0, 1, 2, \ldots \\ T: \text{sampling period} \end{array}$$

Thus: (from sampling Theorem)

$$H(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left(\Omega - \frac{2\pi k}{T}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left(\frac{\omega - 2\pi k}{T}\right) \qquad (*)$$

By generalizing (*) it can be shown that: (ie, place $s = j\Omega$, then $** \to *$)

$$H(z)\Big|_{z=e^{sT}} = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left(s - j\frac{2\pi k}{T}\right) \qquad (**)$$

Consequently, a relation that maps areas of the s-plane to areas of the z-plane is.

$$\boxed{z = e^{sT} \Rightarrow re^{j\omega} = e^{\sigma T} e^{j\Omega T}} \qquad (***)$$

Now, let $\left(H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}\right)$ (easy to obtain with partial fraction expansion)

Then from system theory we know that $h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t} u(t)$

From the impulse invariance constraint $h(n) = h_a(nT) = \sum_{k=1}^{N} A_k e^{s_k nT} u(nT)$

Then: $$H(z) = \sum_{n=0}^{\infty}\sum_{k=1}^{N} A_k e^{s_k nT} z^{-n} = \sum_{k=1}^{N} A_k \sum_{n=0}^{\infty} \left(e^{s_k T}\right)^n z^{-n} = \sum_{k=1}^{N} \frac{A_k}{1 - e^{s_k T} z^{-1}}$$

[From the above we conclude that poles in the s-plane map to poles in the z-plane according to the relation $\underline{z_k = e^{s_k T}}$]

1) Note that zeros might be mapped in a different way than poles. However, this is not a concern in impulse invariance method.

2) In most design applications T is not important so assume $T=1$.

PROPERTIES OF TRANSFORMATION (MAPPING)

From (***) $\quad re^{j\omega} = e^{\sigma T} e^{j\Omega T}$

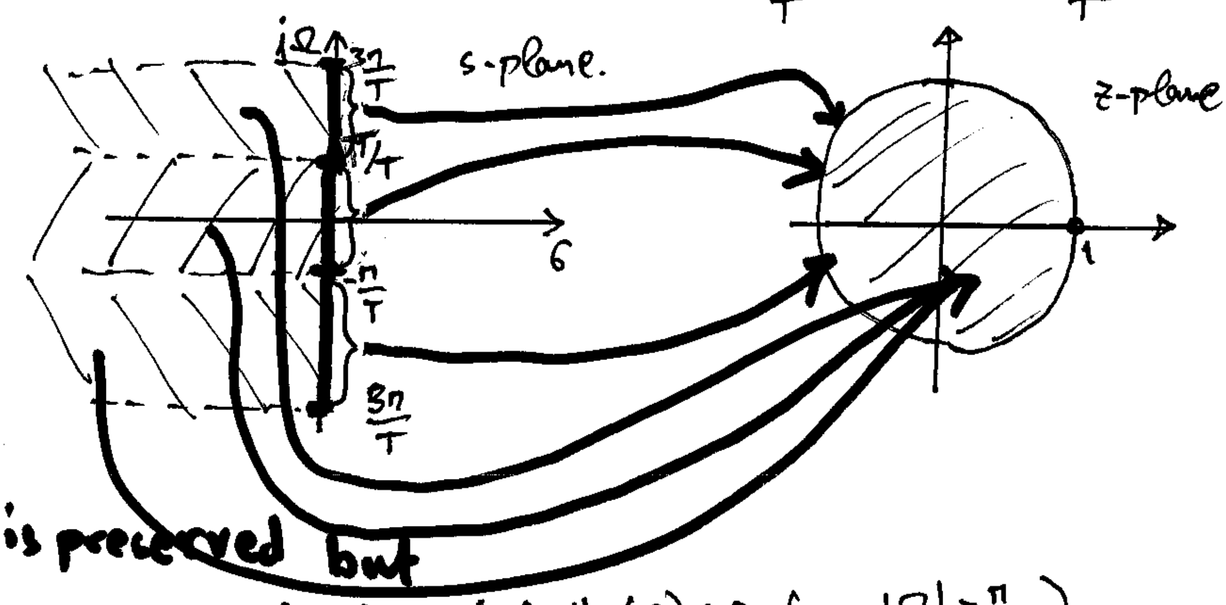
$$\rightarrow \left\{ \begin{array}{l} r = e^{\sigma T} \\ \omega = \Omega T - 2\pi k \end{array} \right\} \quad k = 0, \pm 1, \pm 2, \ldots$$

1) For $\sigma = 0$, $r = 1$ and then $-\pi \le \omega \le \pi$ on the unit circle.

From the equality $\quad \omega = \Omega T - 2\pi k \Rightarrow -\pi \le \Omega T - 2\pi k \le \pi \Rightarrow$

$$\Rightarrow (2k-1)\pi \le \Omega T \le (2k+1)\pi \Rightarrow \frac{(2k-1)\pi}{T} \le \Omega \le \frac{(2k+1)\pi}{T}$$

Illustratively:



*** Shape of spectrum is preceeved but**

*** Aliasing possible** (if $H_a(\Omega) \neq 0$ for $|\Omega| \geq \frac{\pi}{T}$)

2) If $\sigma < 0 \Rightarrow 0 < r < 1 \Rightarrow$ LHP in s-plane $\xrightarrow{\text{maps}}$ inside unit circle i.e., stability is preserved

3) if $\sigma > 0 \Rightarrow r > 1 \Rightarrow$ RHP in s-plane $\xrightarrow{\text{maps}}$ outside unit circle.

DESIGN PROCEDURE with impulse invariance

ECE 431

1) Specifications for digital filter are given in digital domain. for $w = w_p$ and $w = w_s$.

2) Assume T=1. Then design analog filter $H_a(s)$ using frequencies w_p, w_s. (no prewarping is needed)

3) Perform PFE to $H_a(s)$, i.e., $H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$

4) Map poles and coefficients to: $H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{s_k} z^{-1}}$ (T=1)

5) Check design $(H(w) = H(z)|_{z=e^{jw}})$ to determine whether it satisfies given specifications.

EXAMPLE: Design a filter using Impulse invariance method on a Butterworth filter.

(1) Filter specifications.

$w_p = 0.2\pi \qquad 20 \log |H(w_p)| = -1 = R_{pass}$

$w_s = 0.3\pi \qquad 20 \log |H(w_s)| = -15 = R_{stop}$

(2) From notes for Butterworth filter

$$\begin{cases} \log\left[1 + \left(\frac{w_p}{w_c}\right)^{2N}\right] = \frac{|R_p|}{10} \Rightarrow \log\left[1 + \left(\frac{0.2\pi}{w_c}\right)^{2N}\right] = \frac{1}{10} \\ \log\left[1 + \left(\frac{w_s}{w_c}\right)^{2N}\right] = \frac{|R_s|}{10} \Rightarrow \log\left[1 + \left(\frac{0.3\pi}{w_c}\right)^{2N}\right] = \frac{15}{10} \end{cases}$$

$$\begin{cases} 1 + \left(\frac{0.2\pi}{w_c}\right)^{2N} = 10^{0.1} \Rightarrow 2N \log(0.2\pi) - 2N \log w_c = \log[10^{0.1} - 1] \\ 1 + \left(\frac{0.3\pi}{w_c}\right)^{2N} = 10^{1.5} \Rightarrow 2N \log(0.3\pi) - 2N \log w_c = \log[10^{1.5} - 1] \end{cases} \rightarrow w_c = 1.778$$

$$\Rightarrow N = \frac{1}{2} \frac{\log[(10^{1.5}-1)/(10^{0.1}-1)]}{\log[0.3\pi/0.2\pi]} = \frac{1}{2} \frac{2.073}{0.17609} = 5.859224 \rightarrow N = 6$$

→ Then: $$H_a(s) = \frac{(1.778)^6}{\prod_{k=0}^{5}\left(s - 1.778 \cdot e^{j\frac{\pi}{2}} e^{j\frac{(2k+1)\pi}{12}}\right)}$$

$$= \sum_{k=0}^{5} \frac{A_k}{s - \underbrace{1.778 \cdot e^{j\frac{\pi}{2}} e^{j\frac{(2k+1)\pi}{12}}}_{s_k}}$$

→ Then $$H(z) = \sum_{k=0}^{5} \frac{A_k}{s - e^{s_k} z^{-1}} \quad \ldots$$

...

→ Finally, $H(\omega) = H(z)\big|_{z=e^{j\omega}} = \ldots$

and check against problem specifications