(8.2.3) IIR FILTER DESIGN BY THE BILINEAR TRANSFORMATION.

BASIC IDEA: Consider an analog filter with system function

$$\boxed{\frac{Y(s)}{X(s)} = H(s) = \frac{b}{s+a}}$$

This system is characterized by the differential equation: $\frac{dy(t)}{dt} + ay(t) = bx(t)$

- Integrate the differential equation and then use a numerical approximation to the integral.

(1) We can write for $y(t) = \int_{t_0}^{t} y'(\tau)d\tau + y(t_0)$ $\qquad y'(t) = \frac{dy(t)}{dt}$

(2) Appoximate the integral above by the trapezoidal formula at $t = nT$, $t_0 = nT - T$. Thus:

$$y(nT) = \int_{(n-1)T}^{nT} y'(\tau)d\tau + y(nT-T) \cong \frac{T}{2}[y'(nT) + y'(nT-T)] + y(nT-T)$$

(3) $y'(t) + ay(t) = bx(t) \Rightarrow y'(nT) + ay(nT) = bx(nT) \Rightarrow$

$$\Rightarrow y'(nT) = -ay(nT) + bx(nT)$$

Substitute (3) into (2)

From (2) $y(nT) = \frac{T}{2}\left[-ay(nT) + bx(nT) - ay(nT-T) + bx(nT-T)\right] + y(nT-T)$

$\Rightarrow$

$$\Rightarrow \quad y(nT)\left[1+\frac{aT}{2}\right] - y\left(\frac{n-1}{T}\right)\left[1-\frac{aT}{2}\right] = \frac{bT}{2}\left[x(nT) + x(nT-T)\right]$$

$y(\ldots) = y(n)$, $x(nT) = x(n)$ $\Rightarrow$

$$\boxed{\left(1+\frac{aT}{2}\right)y(n) - \left(1-\frac{aT}{2}\right)y(n-1) = \frac{bT}{2}\left[x(n)+x(n-1)\right]}$$

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$\left.\begin{array}{l}\text{Take z-transform}\\ \text{and solve for } H(z)\end{array}\right\}$: $\left(1+\frac{aT}{2}\right)Y(z) - \left(1-\frac{aT}{2}\right)z^{-1}Y(z) = \frac{bT}{2}\left(1+z^{-1}\right)X(z) \Rightarrow$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{(bT/2)(1+z^{-1})}{1+\frac{aT}{2} - \left(1-\frac{aT}{2}\right)z^{-1}} \Rightarrow$$

after simple calculations $\Rightarrow$

$$\boxed{H(z) = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a}}$$

but

$$H(s) = \frac{b}{s+a}$$

Obviously

** The mapping from the s-plane to the z-plane is

$$\boxed{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

billinear Transformation (Holds, in general, for any N-th order differential equation)

and $\boxed{H(z) = H(s)\Big|_{s=2/T\cdot\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}}$

~~Now~~ Solving for z:

$$\boxed{z = \frac{1+(T/2)s}{1-(T/2)s}}$$

Also:

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \overset{z=e^{j\omega}}{=\!=} \frac{2}{T}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right) = \frac{2}{T}\,\frac{e^{-j\frac{\omega}{2}}}{e^{-j\frac{\omega}{2}}}\,\frac{e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}}+e^{-j\frac{\omega}{2}}}$$

$$= \frac{2}{T}\,\frac{j\sin(\omega/2)}{\cos(\omega/2)} = \frac{2}{T}\,j\tan(\omega/2)$$

i.e. $s = \sigma + j\Omega = \frac{2}{T}j\tan\left(\frac{\omega}{2}\right) \Rightarrow$

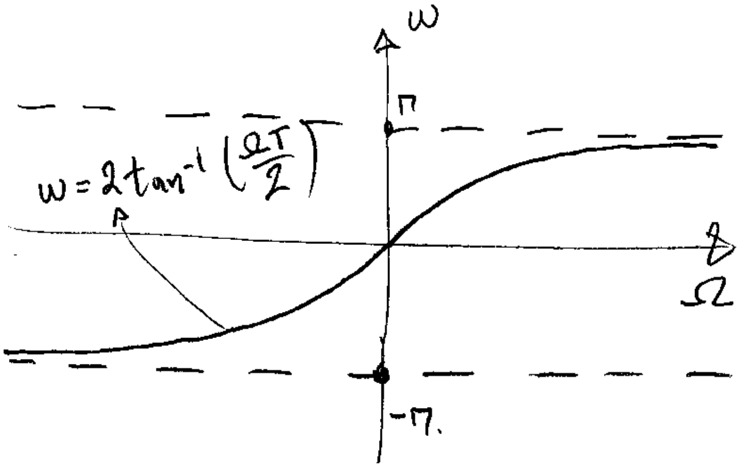
$$\Rightarrow \Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right) \Rightarrow \boxed{\frac{T\Omega}{2} = \tan\left(\frac{\omega}{2}\right)}$$

$\sigma = 0 \Rightarrow$ imaginary axis mapped on unit circle

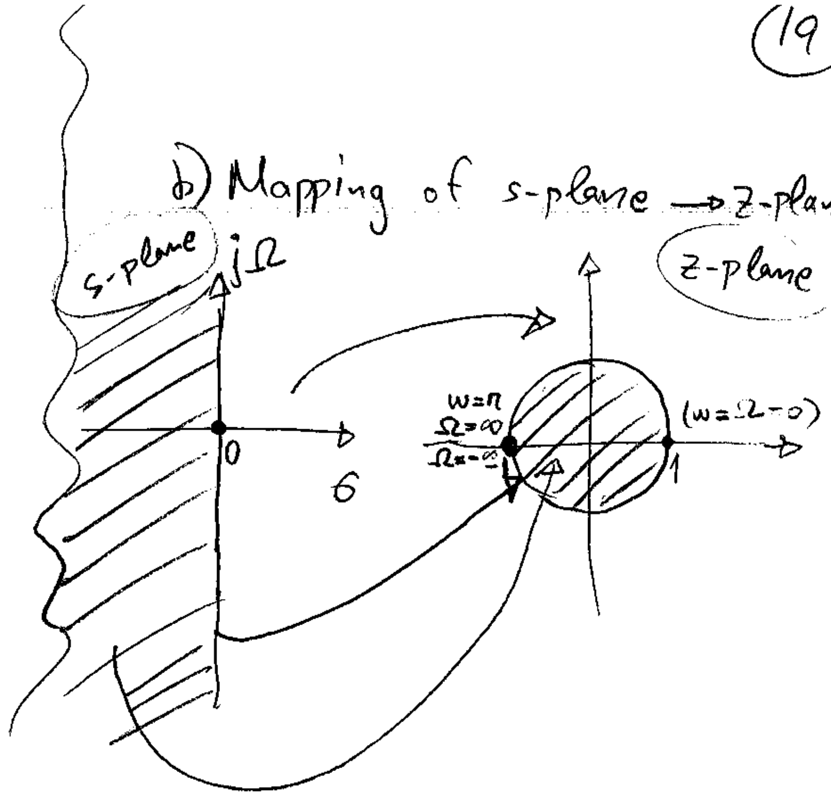
* Since $-\infty < \Omega < \infty$ and $-\pi \le \omega \le \pi$, the entire range in the analog frequency Ω is mapped only <u>once</u> into the range $-\pi \le \omega \le \pi$ of the discrete frequency ω. This frequency compression is called <u>frequency warping</u>.
* It is easy to show if we substitute $z = r \cdot e^{j\omega}$ in the billinear transformation formula that:
  - the LHP (left hand plane $s<0$) in s is mapped inside the unit circle ($|z| < 1$) in z-plane
  - the RHP ($s>0$) in s-plane is mapped outside unit circle ($|z|>1$) in z-plane
  - the point $s = \pm\infty$ is mapped to $z = -1$

Illustratively.

a) Mapping of Ω onto ω. (billinear transformation)



b) Mapping of s-plane → z-plane



Note: From above
- Billinear transformation yields stable digital filters from stable analog filters
- avoids problem of aliasing
- price paid by the distortion in the frequency axis

Note: In most cases the ~~exact~~ value of period T is not important. So usually we use the normalized "warped" frequency

$$\boxed{\omega' = \Omega T = 2\tan\left(\frac{\omega}{2}\right)}$$

III) DESIGN PROCEDURE

1) Specifications of digital filter are given in digital domain for ω_p, ω_s
2) Prewarp specified frequencies $\omega_p' = 2\tan\left(\frac{\omega_p}{2}\right)$, $\omega_s' = 2\tan\left(\frac{\omega_s}{2}\right)$
3) Design analog filter $H(s)$ using the specificat. for ω_p', ω_s'
4) Use billinear transformation with $T=1$ ie $\left[s \to 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]$ to get $H(z)$
5) Check design to see if it satisfies specifications

EXAMPLE Design of filter using Bilinear ~~transform~~ transformation on a Butterworth filter.

Design a digital filter with specifications

$\omega_p = 0.2\pi$, $|H(\omega_p)|^2_{dB} = 20 \log |H(\omega_p)| = -1$

$\omega_s = 0.3\pi$, $|H(\omega_s)|^2_{dB} = 20 \log |H(\omega_s)| = -15$

2) $\omega_p' = 2\tan\left(\frac{\omega_p}{2}\right) = 2\tan\left(\frac{0.2\pi}{2}\right) = 0.65$

$\omega_s' = 2\tan\left(\frac{\omega_s}{2}\right) = 2\tan\left(\frac{0.3\pi}{2}\right) = 1.02$

3) From previous (page 12 of notes) for a Butterworth filter (where $\Omega_p \equiv \omega_p'$, $\Omega_s \equiv \omega_s'$, $\Omega_c = \omega_c'$) we get

$10\log_{10}\frac{1}{1+\left(\frac{\omega_p}{\omega_c}\right)^{2N}} = |R_p|$

$$\begin{cases} \log\left[1+\left(\frac{\omega_p'}{\omega_c'}\right)^{2N}\right] = \frac{|R_p|}{10} \Rightarrow \log\left[1+\left(\frac{0.65}{\omega_c'}\right)^{2N}\right] = \frac{1}{10} \\ \log\left[1+\left(\frac{\omega_s'}{\omega_c'}\right)^{2N}\right] = \frac{|R_s|}{10} \Rightarrow \log\left[1+\left(\frac{1.02}{\omega_c'}\right)^{2N}\right] = \frac{15}{10} \end{cases}$$

$$\Rightarrow \begin{cases} 1+\left(\frac{0.65}{\omega_c'}\right)^{2N} = 10^{0.1} \Rightarrow 2N\cdot\log(0.65) - 2N\cdot\log\omega_c' = \log\left[10^{0.1}-1\right] \\ 1+\left(\frac{1.02}{\omega_c'}\right)^{2N} = 10^{1.5} \Rightarrow 2N\cdot\log(1.02) - 2N\log\omega_c' = \log\left[10^{1.5}-1\right] \end{cases}$$

$$\Rightarrow N = \frac{1}{2}\,\frac{\log\left[(10^{1.5}-1)/(10^{0.1}-1)\right]}{\log\left[1.02/0.65\right]} = 5.30466$$

To meet the specifications we must choose $\boxed{N=6}$

Then one of the above equations (stopband) gives $\boxed{\omega_c' = 0.76622}$

(we do not care which one since aliasing does not occur in bilinear)

(With this ~~value~~ value of ~~ω~~ N and ω_c' the stopband specifications are met exactly, while the passband specifications are exceeded.

1 hws

$$H(s) = \frac{(\omega_c')^6}{\prod_{k=0}^{5} (s-s_k)} = \frac{(0.76622)^6}{\prod_{k=0}^{5}\left(s - \omega_c' e^{j\frac{\pi}{2}} e^{j\frac{(2k+1)\pi}{12}}\right)} = \cdots\cdots$$

④ $$H(z) = H(s)\Big|_{s = 2\frac{1-z^{-1}}{1+z^{-1}}} = \frac{0.20236}{\prod_{k=0}^{5}\left[2\frac{1-z^{-1}}{1+z^{-1}} - 0.76622\, j\, e^{j\frac{(2k+1)\pi}{12}}\right]}$$

after calculations
$\Longrightarrow$

$$H(z) = \frac{0.0007378\,(1+z^{-1})^6}{(1-1.2686\,z^{-1}+0.7051\,z^{-2})(1-1.0106\,z^{-1}+0.3583\,z^{-2})\cdot(1-0.9044\,z^{-1}+0.2155\,z^{-2})}$$

⑤ Implement using cascade or parallel or ... realization.

⑥ Check $H(\omega)$ against desired specifications