8.2 DESIGN OF IIR FILTERS FROM ANALOG FILTERS (8)

TRADITIONAL APPROACH: Take an analog filter and convert it into a digital filter meeting prescribed specifications

Reasoning
- Design of Analog filters: advanced and well developed
- Simple closed form formulas exist in analog filter design $\rightarrow$ simplicity
- Interesting in performance simulation applications (i.e. use digital filters to simulate analog ones)

Analog System function (1) $H_a(s) = \dfrac{B(s)}{A(s)} = \dfrac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} a_k s^k}$

LCCDE (3) $\displaystyle\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$

Laplace transform (2) $H_a(s) = \displaystyle\int_{-\infty}^{\infty} h(t) e^{-st}$

stable if poles in the left half s-plane

Digital System function (1) $H(z) = \dfrac{\sum_{k=0}^{M} d_k z^{-k}}{\sum_{k=0}^{N} c_k z^{-k}}$

LCCDE (3) $\displaystyle\sum_{k=0}^{N} c_k\, y(n-k) = \sum_{k=0}^{M} d_k\, x(n-k)$

Z-transform (2) $H(z) = \displaystyle\sum_{n=-\infty}^{\infty} h(n)\, z^{-n}$

stable if all poles inside unit circle

In the conversion from analog to digital filter

* Imaginary axis ($j\Omega$) of s-plane should ~~must be mapped~~ map onto the unit circle ($|z|=1$) of z-plane so that there will be direct relationship between frequencies in the two domains

** Left-half plane of s-domain should map inside unit circle z-domain and thus stability requirements are preserved

*** Realizable and stable IIR filters cannot have linear phase because

$$\boxed{\begin{array}{l}\text{Linear phase} \Rightarrow H(z) = \pm z^{-N} H(z^{-1}) \\ \Rightarrow \text{if a pole exists inside unit circle} \\ \text{his mirror-image pole should exist outside} \\ \text{unit-circle on contrary to stability requirement}\end{array}}$$

~~8.2.5 Characteristics of Commonly used Analog filters~~

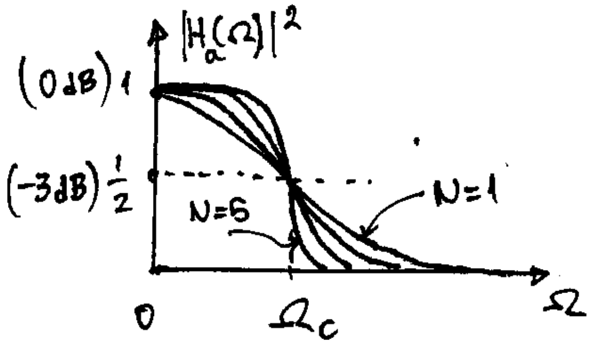
~~Lowpass Butterworth Filters~~

Butterworth analog filters (lowpass filters)

These are Filters with Magnitude-squared frequency response

$$\boxed{|H_a(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}}$$

N-th order Butterworth filter



$$|H_a(\Omega)|^2_{dB} = 10 \log \frac{|H_a(\Omega)|^2}{|H_a(0)|^2} = 10 \log |H_a(\Omega)|^2$$

thus, $|H_a(\Omega_c)|^2_{dB} = -3\text{ dB}$, $|H_a(0)|_{dB} = 0\text{ dB}$

Note that: $\boxed{H_a(\Omega) = H_a(s)\big|_{s=j\Omega}}$

thus, Similarly in s-domain: one can write the relation

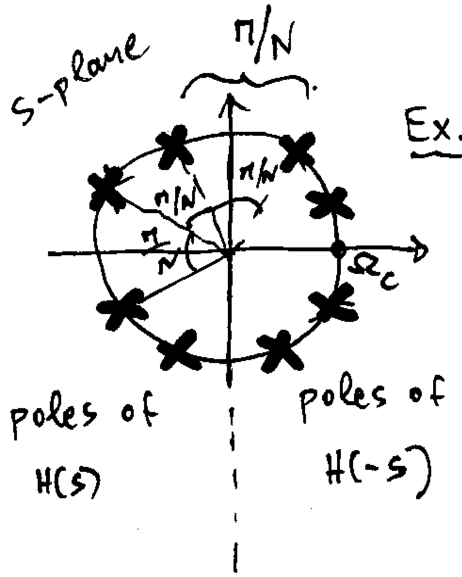
$$H_a(s)\cdot H_a(-s) = \frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}}$$

This expression has poles at:

$$1+\left(\frac{s}{j\Omega_c}\right)^{2N} = 0 \Rightarrow s^{2N} = -(j\Omega_c)^{2N} \Rightarrow s = (-1)^{\frac{1}{2N}} j\Omega_c \Rightarrow$$

$$\Rightarrow s_k = j\Omega_c\, e^{j\frac{(2k+1)\pi}{2N}} = \Omega_c\, e^{j\pi/2} e^{j\frac{(2k+1)\pi}{2N}}$$

$k = 0, 1, \ldots, 2N-1.$



Ex. N=4

* We choose as poles of H(s) $\{s_k\}$, $k=0,1,\ldots,N-1$ in order to satisfy stability criteria.
* Complex conjugate poles $\Longrightarrow$ Butterworth filters have real coefficients.

From previous a stable and causal Butterworth filter of order N, has N-poles (complex conjugate) and thus can be written as

$$H(s) = \frac{A}{(s-s_0)(s-s_1)\cdots(s-s_{N-1})} \qquad (A : \text{gain factor})$$

where $s_0, s_1, \ldots, s_{N-1}$ belong to the left half s-plane.

To calculate A given $\{s_k\}$ one takes into account the complex conjugate property of the poles. Thus

$$H(s)\,H(-s) = \frac{A^2}{(s-s_0)(s-s_0^*)(s-s_1)(s-s_1^*)\cdots\cdots(-s-s_0)(-s-s_0^*)\cdots}$$

$$= \frac{A^2}{(s-s_0)(s-s_0^*)(-s-s_0)(-s-s_0^*)\cdots\cdots(s-s_{\frac{N}{2}-1})\cdots}$$

For $s=0$: $$H(0) = \frac{A^2}{|s_0|^4\,|s_1|^4\cdots|s_{\frac{N}{2}-1}|^4} = \frac{A^2}{\Omega_c^4\,\Omega_c^4\cdots\Omega_c^4} = \frac{A^2}{\Omega_c^{2N}} = 1$$

$$\Rightarrow \boxed{A = \Omega_c^N}$$

Then,

$$\boxed{H_a(s) = \frac{\Omega_c^N}{\prod_{k=0}^{N-1}(s-s_k)}}$$

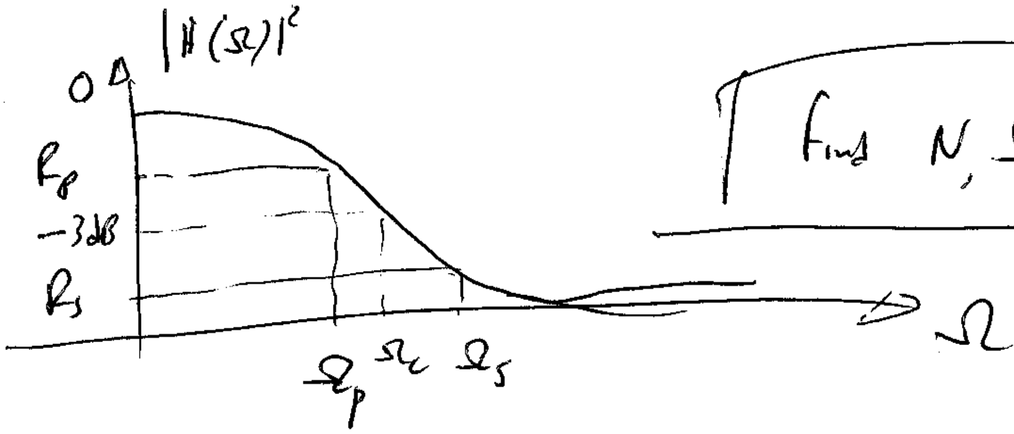
If N is odd then there is a pole at $s=-\Omega_c$ otherwise in complex conjugate terms

Now assuming the following specifications for the Butterworth filter:

Given

R_p, Ω_p

R_s, Ω_s



Find N, Ω_c

From above

$$|H(\Omega)|^2_{dB} = 10\log\frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \geq R_p \quad , \Omega \leq \Omega_p$$

$$\Rightarrow -10\log\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right] \geq R_p \Rightarrow \log\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right] \leq \frac{|R_p|}{10}$$

Similarly:

$$|H(\Omega)|^2_{dB} = 10\log\frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \leq R_s \quad \Omega \geq \Omega_s$$

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To find N, Ω_c solve the two equations

$$\boxed{\begin{aligned} \log\left[1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}\right] &= \frac{|R_p|}{10} \\ \log\left(1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}\right] &= \left|\frac{R_s}{10}\right| \end{aligned}}$$