THE CONCEPT OF FREQUENCY

A) SINUSOIDS

- **Continuous time** $(-\infty < t < \infty)$
- $x(t) = A\cos[\Omega t + \theta] = A\cos[2\pi F t + \theta]$
- F: cycles/sec
- Periodic with period $T = \frac{1}{F}$, i.e.,
  $x(t) = x(t+T) = x(t+\ell T)$, T: fundamental period
- F takes any value: $-\infty < F < \infty$
- Distinct sinusoids are obtained from any distinct frequencies in $-\infty < F < \infty$
- The highest rate of oscillation is achieved when $\Omega \to \infty$, or $F \to \infty$

- **Discrete time** $(n = 0, \pm 1, \pm 2, \ldots$
- $x(n) = A\cos[\omega n + \theta] = A\cos[2\pi f n + \theta]$
- f: cycles/sample
- Periodic with period N samples, i.e.
  $x(n) = x(n+N) = x(n+\ell N)$
  N: fundamental period

$A\cos(\omega n + \theta) = A\cos(\omega(n+N) + \theta) \Rightarrow$

iff

$\Rightarrow \omega n = \omega(n+N) \pm 2k\pi \Rightarrow \boxed{f = \pm\frac{k}{N}}$

i.e., f must be rational

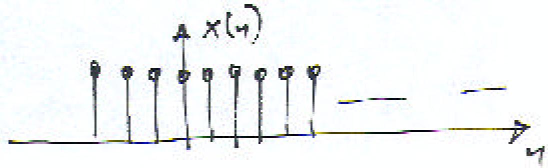
- Let $\omega_1 = \omega_2 + 2k\pi$: Then $\cos[\omega_1 n + \theta] = \cos[\omega_2 n + \theta]$
  Thus distinct discrete sinusoids can be obtained only in intervals of 2π, i.e. $-\pi \le \omega \le \pi$ or $0 \le \omega \le 2\pi$.
  (principal range) i.e. $-\frac{1}{2} \le f \le \frac{1}{2}$ or $0 \le f \le 1$
- The highest rate of oscillation for a discrete sinusoid is achieved when $\omega = \pi$ or $f = \frac{1}{2}$.

Ex. Let $x(n) = \cos \omega_0 n$

$\omega_0 = 0, \pi/8, \pi/4, \pi/2$

$f = 0, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}$

a) $\omega_0 = 0$, $f = 0$, period $N = \infty$



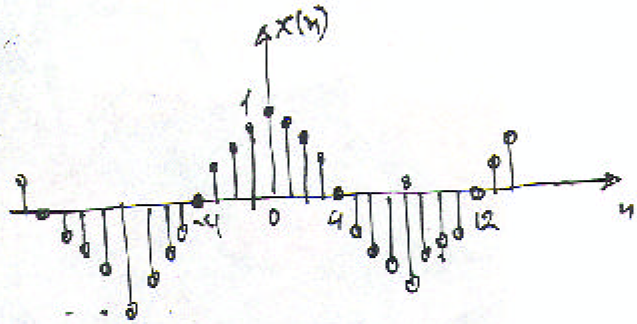
b) $\omega_0 = \frac{\pi}{8}$, $f = \frac{1}{16}$, period $N = 16$.

zeros

$\cos\left[\frac{\pi n}{8}\right] = 0 \Rightarrow \frac{\pi n}{8} = k\pi + \frac{\pi}{2} \Rightarrow n = 8k + 4$

$k = 0, 1, 2, 3, \ldots$

$n = 4, 12, 20, \ldots$



maxima

$\cos\left[\frac{\pi n}{8}\right] = \pm 1 \Rightarrow \frac{\pi n}{8} = k\pi \Rightarrow n = 8k$

c) ...

Highest possible rate of oscillation

d) $\omega_0 = \pi$, $f = 1/2$, $N = 2$

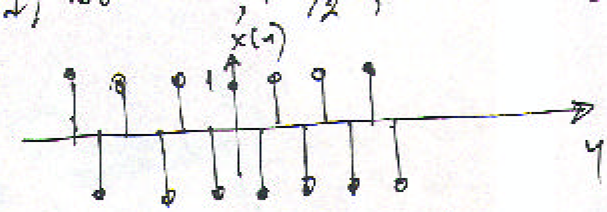
zeros

$\cos[\pi n] = 0$

$\pi n = k\pi + \pi/2 \Rightarrow n = k + 1/2$

n : integer

no solution



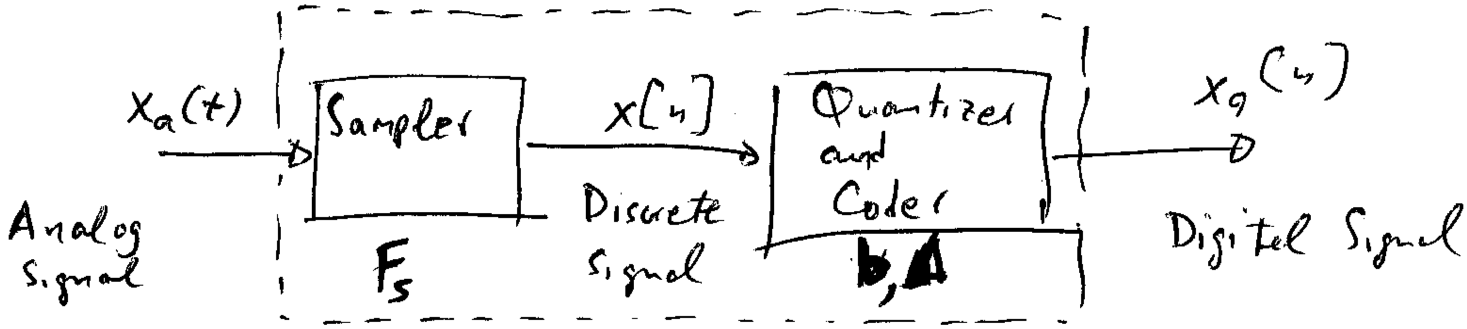
maxima $\pi n = k\pi \Rightarrow n = k$

ECE 431, Lecture 2

1.

Analog to Digital Conversion (ADC)

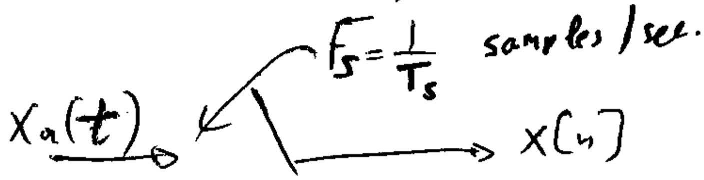
* Necessary in order to use digital processing methods on "real world signals"

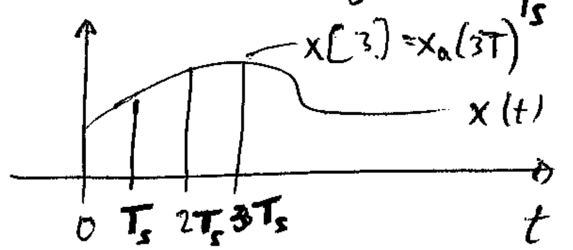


A/D (or ADC)

IDEAL SAMPLING: [Uniform, i.e. take one sample every T_s seconds]

$x[n] = x_a(nT)$, $-\infty < n < \infty$, $\left[\begin{array}{l}\text{Sampling Period } T_s \\ \text{" frequency } F_s = \frac{1}{T_s}\end{array}\right]$





* Let $x_a(t) = A\cos(\overbrace{2\pi F}^{\Omega} t + \theta)$

Then $x[n] = x_a(nT_s) = A\cos(2\pi F n T_s + \theta) = A\cos((\underbrace{2\pi F T_s}_{\omega}) n + \theta)$

$= A\cos(\omega n + \theta)$

ω: Normalized frequency (discrete domain frequency) $\boxed{\omega = \Omega T_s}$ rad/~~sample~~ (true frequency: Ω) or $\boxed{f = \frac{F}{F_s} = \frac{\omega}{2\pi}}$ cycles/sample

* $|F| < \infty$, $|f| \leq \frac{1}{2}$ (Can you figure out why?)
* f must be rational in order for $x(n)$ to be periodic $(f = \frac{k}{N})$

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Also:

$$x_a(nT_s) = A\cos(\Omega n T_s + \theta) = A\cos\left(\left(\Omega + \frac{2\pi k}{T_s}\right) n T_s + \theta\right)$$ periodic in Ω with period $2\pi/T_s$

$$x[n] = A\cos(\omega n + \theta) = A\cos((\omega + 2\pi k) n + \theta)$$: periodic in ω with period 2π

Now let $x(t)$ be a bandlimited analog signal
Consider the following modeling of the sampling process

$$x_a(t) \rightarrow \otimes \xrightarrow{x_s(t) = \sum_{l=-\infty}^{\infty} x_a(t)\delta(t - lT_s)} \boxed{\text{Convert to discrete time sequence}} \xrightarrow{x[n] = \sum_{l=-\infty}^{\infty} x_a(lT)\delta(n-l)}$$

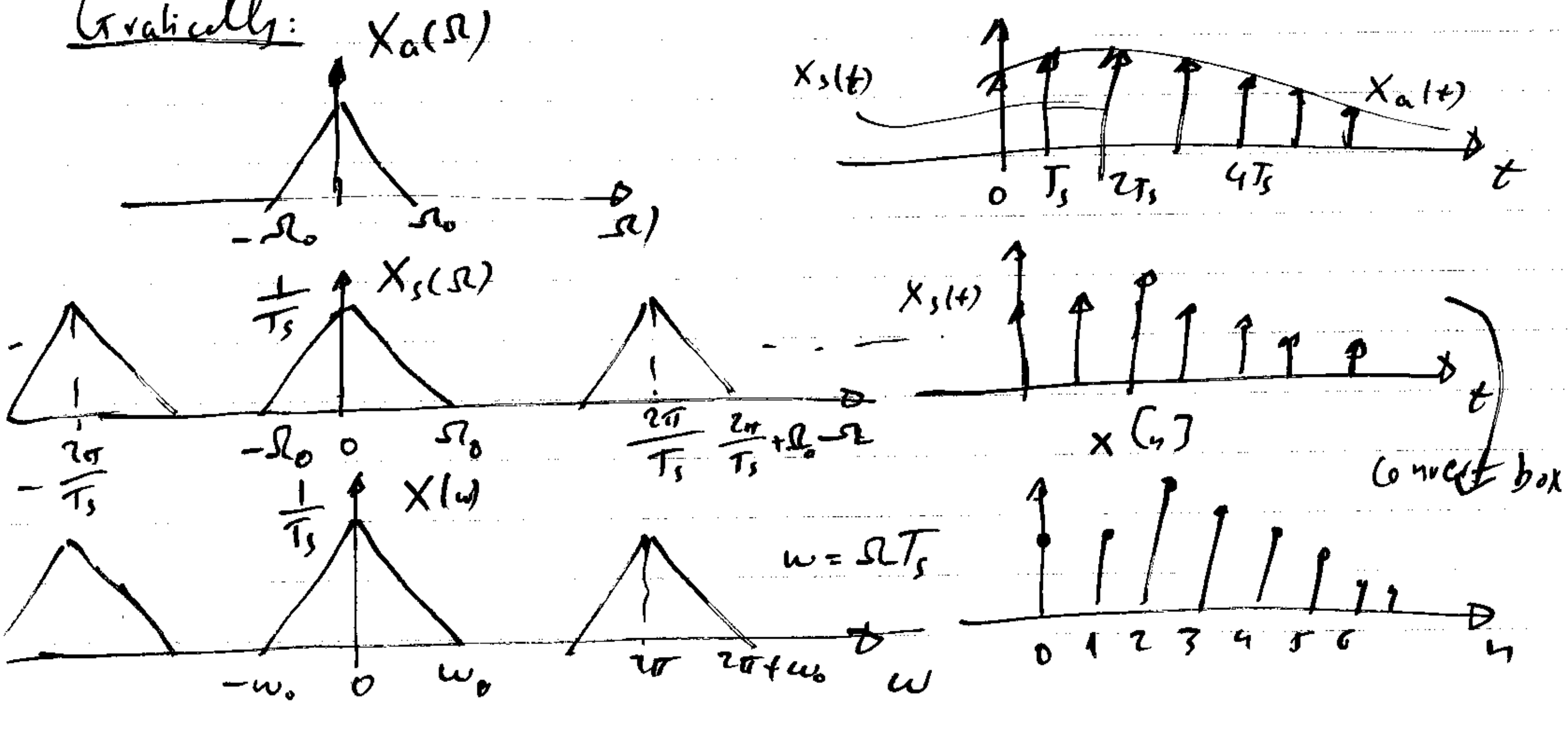
$$s(t) = \sum_{l=-\infty}^{\infty} \delta(t - lT_s) \ \underset{\text{expansion}}{\overset{\text{Fourier Series}}{=\!=}} \ \sum_{l=-\infty}^{\infty} \frac{1}{T_s} e^{+j k \frac{2\pi}{T_s} t}$$

FS coefficient $\left(a_k = \frac{1}{T_s}\int_{T_s} s(t) e^{-j\frac{2\pi k}{T_s}t} = \frac{1}{T_s}\right)$

Therefore note:

$$x_s(t) = \frac{1}{T_s}\sum_{l=-\infty}^{\infty} x_a(t) e^{jk\frac{2\pi}{T_s}t} \ \xleftrightarrow[\text{Transform}]{\text{Fourier}} \ X_s(\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a\left(\Omega + \frac{2\pi k}{T_s}\right)$$

Graphically:



→ $X(\Omega)$ must be bandlimited ($X(\Omega)=0$ $|\Omega|>\Omega_0$)

→ To avoid distortion from overlapping among replicas of the spectrum (named "aliasing") $\boxed{\Omega_s \geq 2\Omega_0}$

→ "Nyquist rate": $\Omega_s = 2\Omega_0$ (given Ω_0)
→ "Nyquist frequency": $\Omega_0 = \frac{\Omega_s}{2}$ (given Ω_s)

* Aliasing is non-reversible non-recoverable distortion

Assuming no aliasing occurs then $x_a(t)$ can be recovered (completely reconstructed) from $x[n]$ by means of ideal low pass (LP) filtering.