

LECTURES ~~1, 2~~ ~~3, 4, 5, 6~~ ~~7, 8, 9, 10, 11, 12, 13, 14, 15, 16~~

(1)

MULTIRATE SYSTEMS

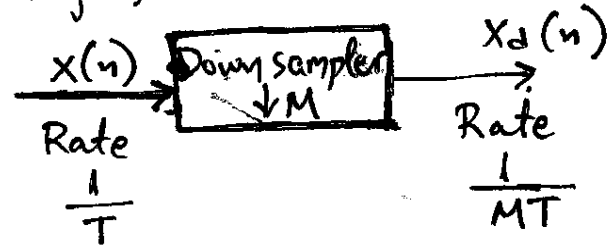
"The sampling rate is not constant throughout the system"

Decimation: $\left\{ \begin{array}{l} \text{filtering +} \\ \text{down-sampling operation} \end{array} \right.$

Interpolation: up-sampling operation + filtering

● M-fold decimation (M integer).

Let $x_d(n) = x(Mn)$



Let $x(n) = X_a(nT)$ then $x_d(n) = x(Mn) = X_a(nMT)$

From sampling theorem

$$X(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\left(\frac{\omega - 2\pi k}{T}\right) \quad (*)$$

and
$$X_d(\omega) = \frac{1}{MT} \sum_{l=-\infty}^{\infty} X_a\left(\frac{\omega - 2\pi l}{MT}\right) = \frac{1}{MT} \sum_{l=Mr+i} X_a\left(\frac{\omega - 2\pi(Mr+i)}{MT}\right)$$

where $r = 0, \pm 1, \pm 2, \dots$, $i = 0, 1, \dots, M-1$.

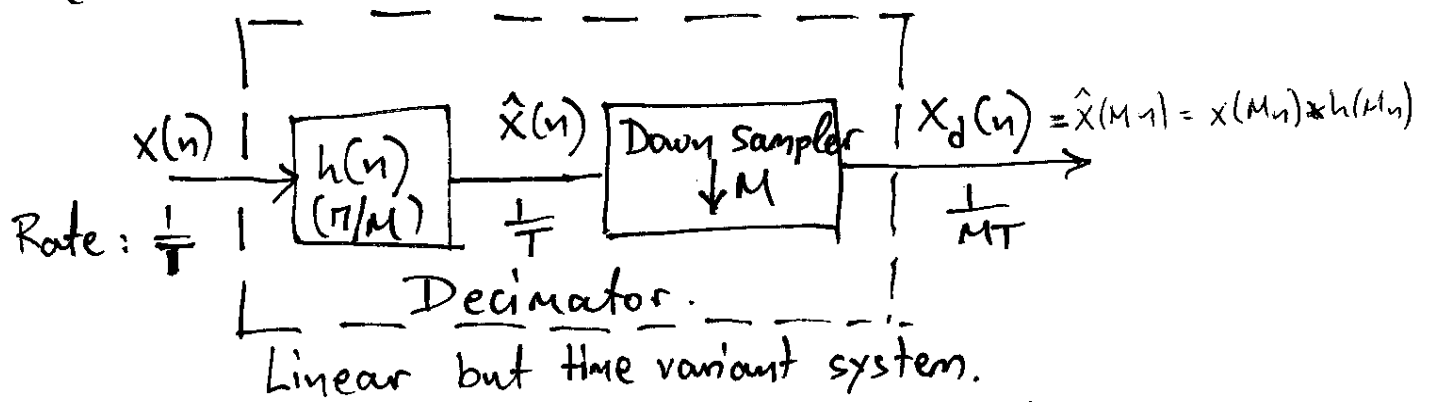
$$\begin{aligned} \Rightarrow X_d(\omega) &= \frac{1}{MT} \sum_{r=-\infty}^{\infty} \sum_{i=0}^{M-1} X_a\left(\frac{\omega - 2\pi i}{MT} - \frac{2\pi r}{T}\right) = \\ &= \frac{1}{M} \sum_{i=0}^{M-1} \left[\frac{1}{T} \sum_{r=-\infty}^{\infty} X_a\left(\frac{\omega - 2\pi i}{MT} - \frac{2\pi r}{T}\right) \right] \Rightarrow (*) \end{aligned}$$

$$X_d(w) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(\frac{w-2\pi i}{M}\right) \iff X_d(n) = X(Mn)$$

Expansion + aliasing $\iff$ Compression

* To avoid aliasing $X(w)$ must be bandlimited in the range $-\frac{\pi}{M} \leq w \leq \frac{\pi}{M}$ (lowpass signal)

Otherwise a filter should be placed before the down-sampler.
(decimation filter)

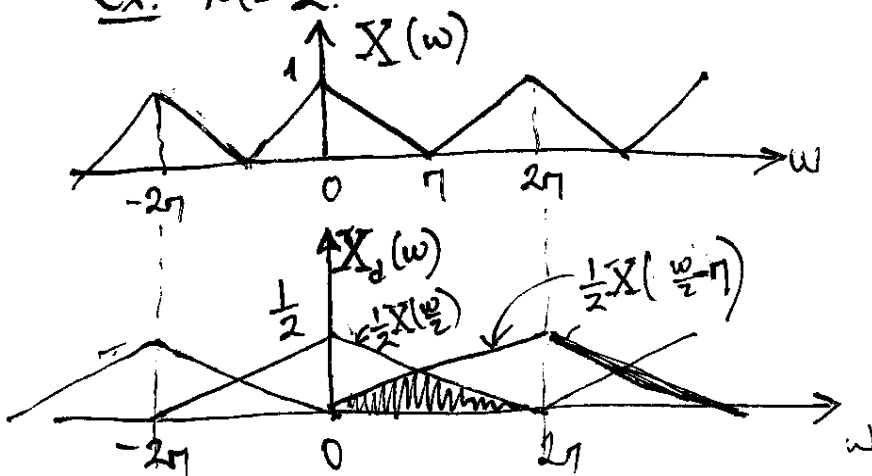


In this case:
$$X_d(w) = \frac{1}{M} \sum_{i=0}^{M-1} \hat{X}\left(\frac{w-2\pi i}{M}\right) = \frac{1}{M} \sum_{i=0}^{M-1} H\left(\frac{w-2\pi i}{M}\right) X\left(\frac{w-2\pi i}{M}\right)$$

assuming no aliasing $= \frac{1}{M} H\left(\frac{w}{M}\right) X\left(\frac{w}{M}\right)$, $-\pi \leq w \leq \pi$

Ex: $M=2$

$$X_d(w) = \frac{1}{2} \left[X\left(\frac{w}{2}\right) + X\left(\frac{w}{2} - \pi\right) \right]$$

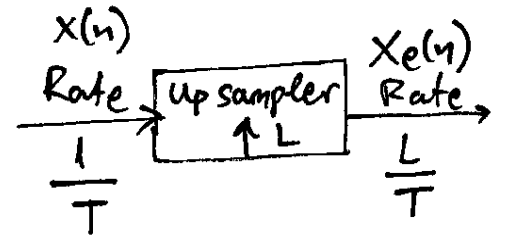


Aliasing would be avoided if we apply a L.P. filter of bandwidth $0 < |w| < \frac{\pi}{2}$. But then distortion is introduced.

(3)

Interpolation by a factor L (L : integer)

Let $x_e(n) = \begin{cases} x(\frac{n}{L}) & , n=0, \pm L, \pm 2L \dots \\ 0 & \text{otherwise} \end{cases}$ (L-1 zeros are inserted after each sample of $x(n)$)



In this case $x_e(n) = \sum_{k=-\infty}^{\infty} x(k) \delta(n - kL) =$

$$\begin{aligned} \text{Thus, } X_e(\omega) &= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k) \delta(n - kL) \right] e^{-j\omega n} = \\ &= \sum_k x(k) \cdot \sum_n e^{-j\omega n} \delta(n - kL) = \sum_k x(k) e^{-j\omega kL} \end{aligned}$$

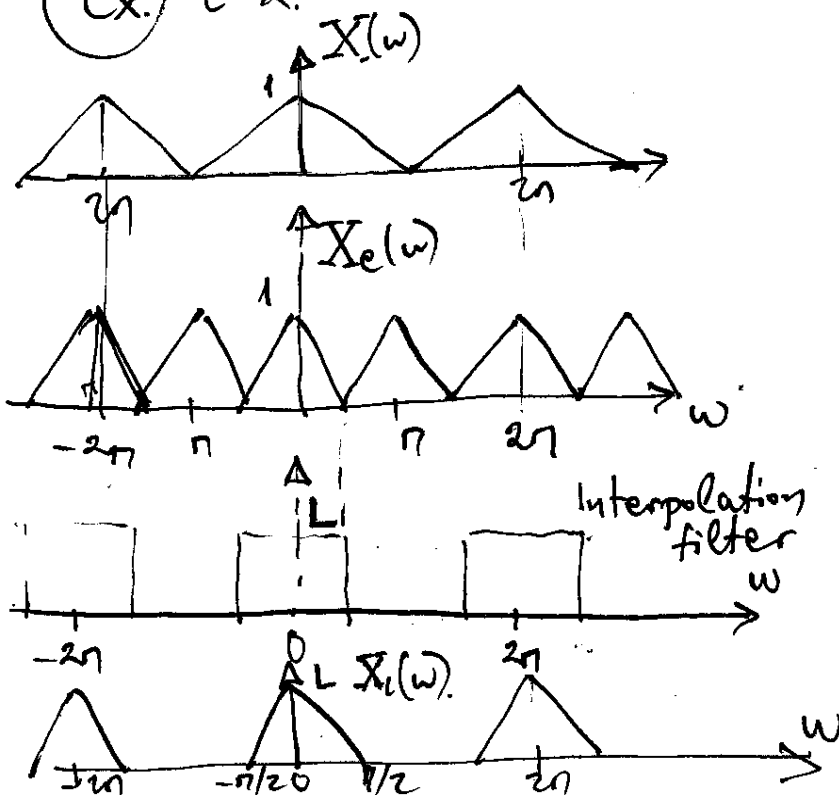
$$\Rightarrow X_e(\omega) = X(\omega L)$$

Compression + Imaging

in N-DFT terms: $\begin{bmatrix} \tilde{X}(n) R_N(n) \xleftrightarrow{\text{N-DFT}} \tilde{X}(k) R_N(k) \\ \tilde{X}_e(n) R_{NL}(n) \xleftrightarrow{\text{NL-DFT}} \tilde{X}(k) R_{NL}(k) \end{bmatrix}$

(L-1 images of $X(\omega)$ are inserted into one period of $X_e(\omega)$)

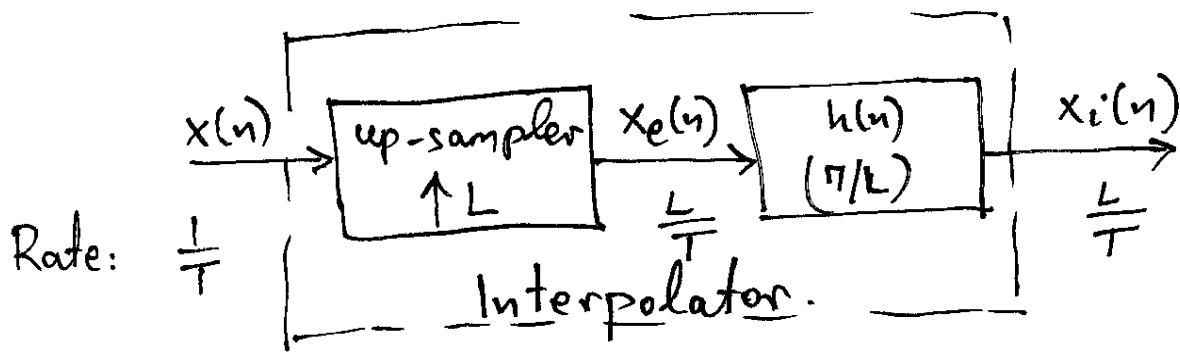
Ex. $L=2$



If we want to reject the (L-1) images we must filter $x_e(n)$ with a low-pass or in general (band pass) filter of gain L

and bandlimited to $0 \leq |\omega| \leq \frac{\pi}{L}$. Then, $\frac{L-1}{L}$ non zero values are interpolated after each value of $x(n)$.

(4)



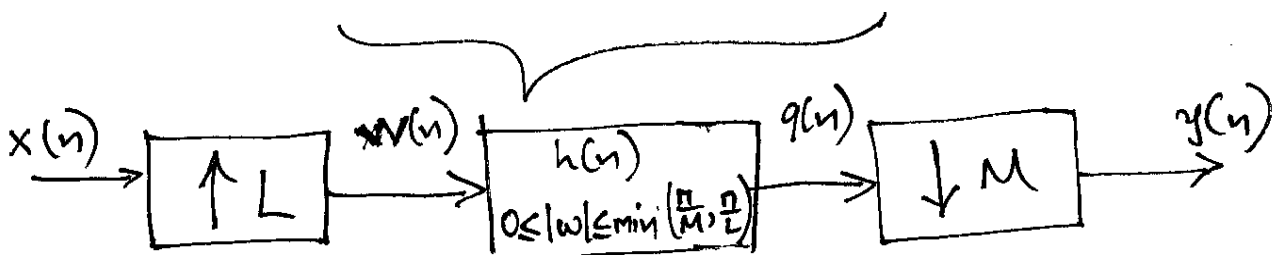
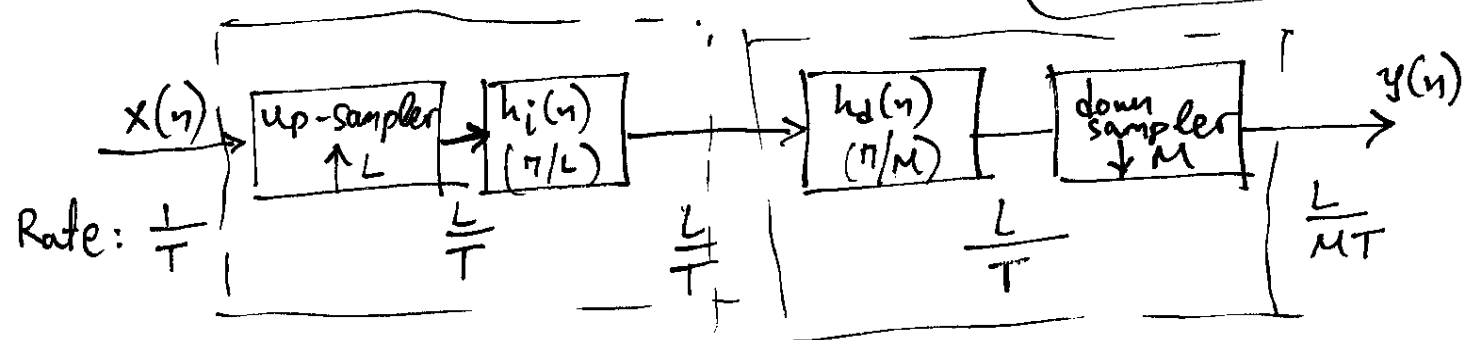
In this case: $x_i(n) = h(n) * x_e(n) = \sum_k h(n-k) x_e(k)$

$$= \sum_k h(n-k) \sum_l x_e(l) \delta(k-lL) =$$

$$= \sum_{l=-\infty}^{\infty} x_e(l) h(n-lL)$$

Also, $X_i(\omega) = H(\omega) \cdot X_e(\omega) = H(\omega) \cdot X(L\omega)$

● Change of sampling rate by $\frac{L}{M}$: noninteger



$$q(n) = \sum_k x(k) h(n-kL) \Rightarrow y(n) = \sum_k x(k) h(Mn-kL)$$

$$Q(\omega) = H(\omega) \cdot X(\omega L) \Rightarrow Y(\omega) = \frac{1}{M} \sum_{i=0}^{M-1} H\left(\frac{\omega - 2\pi i}{M}\right) \cdot X\left(\frac{\omega - 2\pi i}{M} L\right)$$

Assuming no aliasing $Y(\omega) = \frac{1}{M} \cdot H(\omega/M) X(\omega L/M) \quad -\pi \leq \omega \leq \pi$