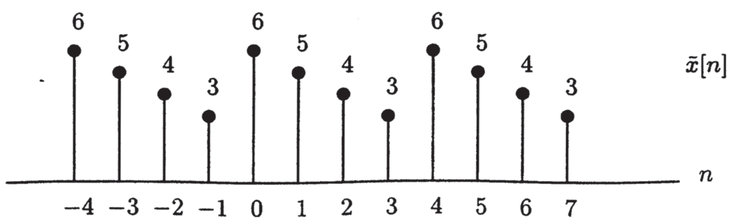
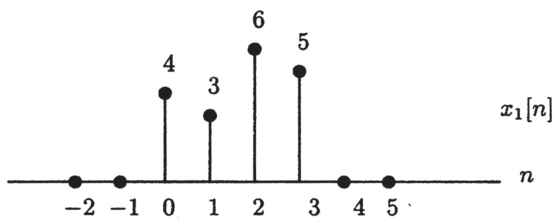
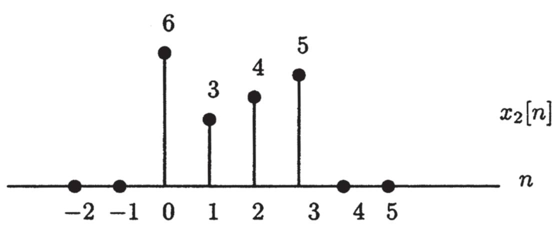
8.24. We may approach this problem in two ways. First, the notion of modulo arithmetic may be simplified if we utilize the implied periodic extension. That is, we redraw the original signal as if it were periodic with period $N = 4$. A few periods are sufficient:



To obtain $x_1[n] = x[((n-2))_4]$, we shift by two (to the right) and only keep those points which lie in the original domain of the signal (i.e. $0 \le n \le 3$):



To obtain $x_2[n] = x[((-n))_4]$, we fold the pseudo-periodic version of $x[n]$ over the origin (time-reversal), and again we set all points outside $0 \le n \le 3$ equal to zero. Hence,



Note that $x[((0))_4] = x[0]$, etc.

In the second approach, we work with the given signal. The signal is confined to $0 \le n \le 3$; therefore, the circular nature must be maintained by picturing the signal on the circumference of a cylinder.

8.25. Problem 1 in Spring 2003 Final exam.
Appears in: Spring05 PS8, Spring04 PS7.

Problem

Consider the signal $x[n] = \delta[n-4] + 2\delta[n-5] + \delta[n-6]$.

(a) Find $X(e^{j\omega})$ the discrete-time Fourier transform of $x[n]$. Write expressions for the magnitude and phase of $X(e^{j\omega})$, and sketch these functions.

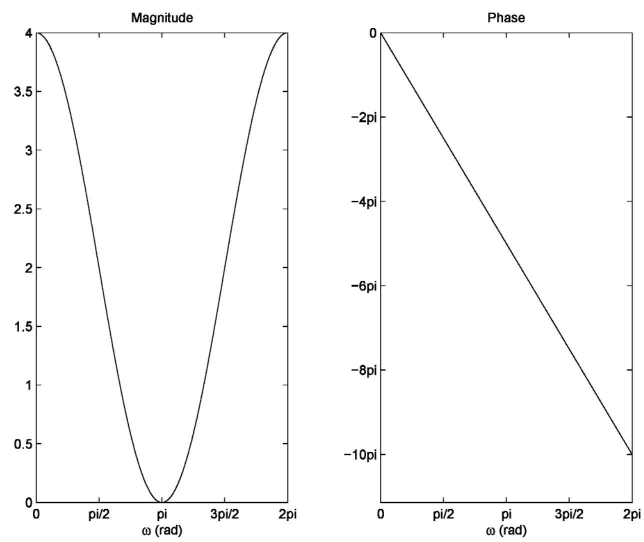
(b) Find all values of N for which the N-point DFT is a set of real numbers.

(c) Can you find a 3-point causal signal $x_1[n]$ (i.e., $x_1[n] = 0$ for $n < 0$) for which the 3-point DFT of $x_1[n]$ is:

$$X_1[k] = |X[k]| \qquad k = 0, 1, 2$$

Solution from Spring05 PS8

(a) $X(e^{j\omega}) = e^{-j4\omega} + 2e^{-j5\omega} + e^{-j6\omega} = e^{-j5\omega}2(1 + cos(\omega))$.
$|X(e^{j\omega})| = |2(1 + cos(\omega))| = 2(1 + cos(\omega))$, and $\angle(X(e^{j\omega})) = -5\omega$.

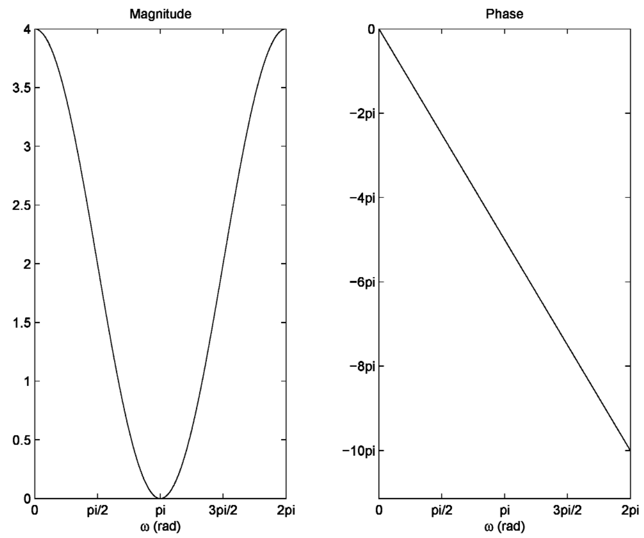


(b) The N-point DFT of $x[n]$ is the set of N samples of $X(e^{j\omega})$, at $\omega_k = 2\pi k/N$, $k = 0, 1, .., N-1$. The set of frequencies where $X(e^{j\omega})$ is purely real is $\omega = 2\pi k/10$, $k = 0, 1, .., 9$ (we need $\angle X(e^{j\omega}) = 5\omega = \pi M$, for some integer M). That means that the DFT will be a real sequence for $N = 1, 2, 5, 10$.

(c) The three-point time-aliased version of $x[n]$ is the sequence $\delta[n] + \delta[n-1] + 2\delta[n-2]$. The sequence of absolute values of its 3-point DFT is $|X_3[k]| = 4\delta[k] + \delta[k-1] + \delta[k-2]$. The inverse 3-point DFT of this sequence is the desired sequence, $x_1[n] = 2\delta[n] + \delta[n-1] + \delta[n-2]$. Note that $x_1[n]$ replicated is a real and even signal so that the frequency domain is also real and even.

Solution from Spring04 PS7

(a) $X(e^{j\omega}) = e^{-j4\omega} + 2e^{-j5\omega} + e^{-j6\omega} = e^{-j5\omega}2(1 + cos(\omega))$.
$|X(e^{j\omega})| = |2(1 + cos(\omega))| = 2(1 + cos(\omega))$, and $\angle(X(e^{j\omega})) = -5\omega$.

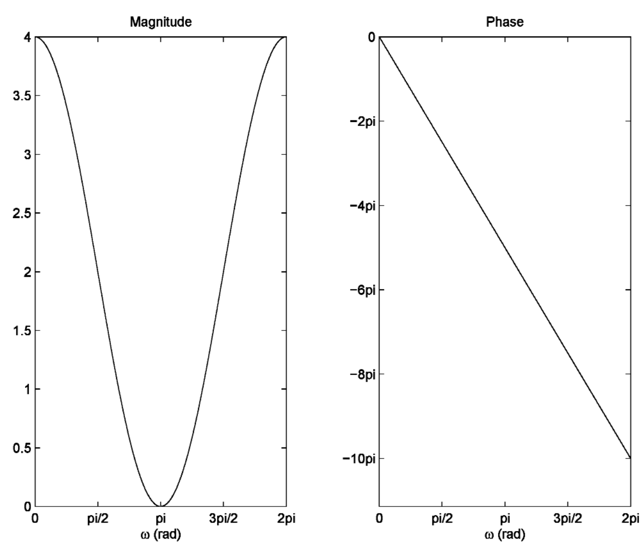


(b) The N-point DFT of $x[n]$ is the set of N samples of $X(e^{j\omega})$, at $\omega_k = 2\pi k/N, \quad k = 0, 1, .., N-1$. The set of frequencies where $X(e^{j\omega})$ is purely real is $\omega = 2\pi k/10$, $k = 0, 1, .., 9$ (we need $\angle X(e^{j\omega}) = 5\omega = \pi M$, for some integer M). That means that the DFT will be a real sequence for $N = 1, 2, 5, 10$.

(c) The three-point time-aliased version of $x[n]$ is the sequence $\delta[n] + \delta[n-1] + 2\delta[n-2]$. The sequence of absolute values of its 3-point DFT is $|X_3[k]| = 4\delta[k] + \delta[k-1] + \delta[k-2]$. The inverse 3-point DFT of this sequence is the desired sequence, $x_1[n] = 2\delta[n] + \delta[n-1] + \delta[n-2]$.

Solution from Spring03 Final

(a) $X(e^{j\omega}) = e^{-j4\omega} + 2e^{-j5\omega} + e^{-j6\omega}$



(b) $N = 5, 10$

(c) $x_1[n] = 2\delta[n] + \delta[n-1] + \delta[n-2]$

8.26. A. We know

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j\frac{\pi}{N}n} e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j\frac{\pi}{N}n(1+2k)}$$

Let $\omega_k = \dfrac{\pi(1+2k)}{N}$. Then

$$\tilde{X}[k] = \sum_{n=0}^{N-1} x[n] e^{-j\omega_k n} = X\left(e^{j\omega_k}\right)$$

$$= X\left(e^{j\left(\frac{\pi+2\pi k}{N}\right)}\right), \quad k = 0,1,\ldots,N-1.$$

B. The frequencies of sampling are given by

$$\omega_k = \frac{\pi(1+2k)}{N}, \quad k = 0,1,\ldots,N-1.$$

C. Given the modified $\tilde{X}[k]$, we can use the inverse transform to find $\tilde{x}[n]$. To get $x[n]$ from $\tilde{x}[n]$ it is a simple point-by-point multiplication given by

$$x[n] = e^{j\frac{\pi}{N}n} \tilde{x}[n].$$

8.27. I. If $G[k]=10\delta[k]$, then $g[n]=1, \quad n=0,\ldots,9$. We can now find $G(e^{j\omega})$ as

$$\begin{aligned} G(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n} \\ &= \sum_{n=0}^{9} e^{-j\omega n} \\ &= \frac{1-e^{-j10\omega}}{1-e^{-j\omega}} \\ &= e^{-j\frac{9}{2}\omega}\frac{\sin(5\omega)}{\sin(\omega/2)}. \end{aligned}$$

8.28. **(a)** Using the analysis equation

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$
$$= \sum_{n=0}^{5} x[n] W_6^{kn}$$
$$= 6W_6^0 + 5W_6^k + 4W_6^{2k} + 3W_6^{3k} + 2W_6^{4k} + W_6^{5k}.$$

(b)

$$W[k] = W_6^{-2k} X[k]$$
$$= 6W_6^{-2k} + 5W_6^{-k} + 4 + 3W_6^k + 2W_6^{2k} + W_6^{3k}.$$

Using the fact that $W_6^k = e^{-j\frac{2\pi k}{6}}$,

$$W_6^{-2k} = e^{j\frac{4\pi k}{6}} = e^{j\frac{4\pi k}{6}} \times e^{-j2\pi k} \text{ (since } e^{-j2\pi k} = 1)$$
$$= e^{-j\frac{8\pi k}{6}} = W_6^{4k},$$

and similarly

$$W_6^{-k} = W_6^{5k}.$$

Then

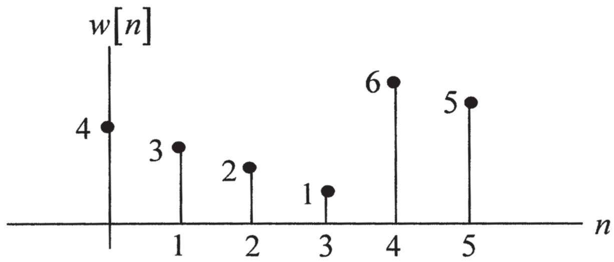
$$W[k] = 4 + 3W_6^k + 2W_6^{2k} + W_6^{3k} + 6W_6^{4k} + 5W_6^{5k}.$$

Using the synthesis equation,

$$w[n] = \frac{1}{6}\sum_{k=0}^{5} W[k] W_6^{-kn}.$$

We could go ahead and solve the problem in this "brute force" method, but notice that each $\delta[n-k] \xrightarrow{DFT} W_N^k$. Then,

$$w[n] = 4\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3] + 6\delta[n-4] + 5\delta[n-5].$$



Notice that multiplying by W_6^{-2k} in frequency has the effect of a shift of 2 in time, but modulo 6.

(c) One way to do this is to compute the linear convolution and then add copies of it shifted by N (6 in this case). Another method is to use the DFT, find the product $H[k]X[k]$, and then take an inverse DFT. We know

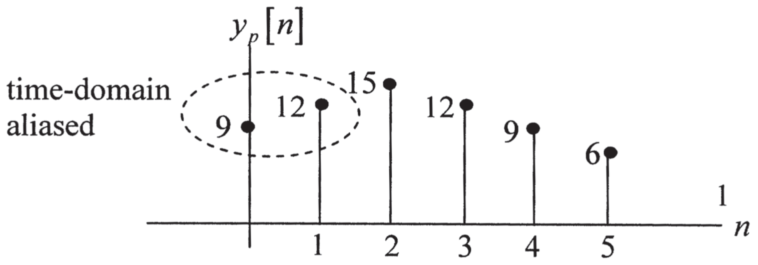
$$X[k] = 6W_6^0 + 5W_6^k + 4W_6^{2k} + 3W_6^{3k} + 2W_6^{4k} + W_6^{5k}$$
$$H[k] = 1 + W_6^k + W_6^{2k}$$

Then

$$\begin{aligned} Y_p[k] = 6 + 5W_6^k + 4W_6^{2k} + 3W_6^{3k} + 2W_6^{4k} + W_6^{5k} \\ + 6W_6^k + 5W_6^{2k} + 4W_6^{3k} + 3W_6^{4k} + 2W_6^{5k} + W_6^{6k} \\ + 6W_6^{2k} + 5W_6^{3k} + 4W_6^{4k} + 3W_6^{5k} + 2W_6^{6k} + W_6^{7k} \\ = 9 + 12W_6^k + 15W_6^{2k} + 12W_6^{3k} + 9W_6^{4k} + 6W_6^{5k}, \end{aligned}$$

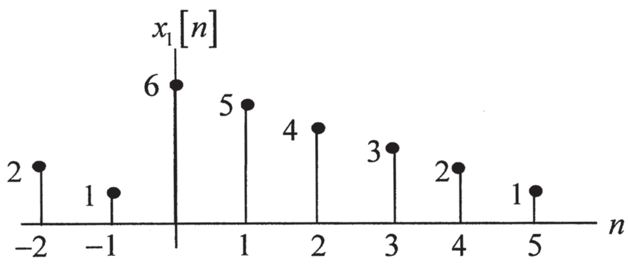
where we have used $W_6^{6k} = 1$ and $W_6^{7k} = W_6^k$. Now we have

$$y_p[n] = 9\delta[n] + 12\delta[n-1] + 15\delta[n-2] + 12\delta[n-3] + 9\delta[n-4] + 6\delta[n-5].$$



(d) To ensure that no time-domain aliasing occurs in the output, N should be large enough to accommodate the length of the linear convolution. That is, $N \geq 6 + 3 - 1 = 8$

(e)

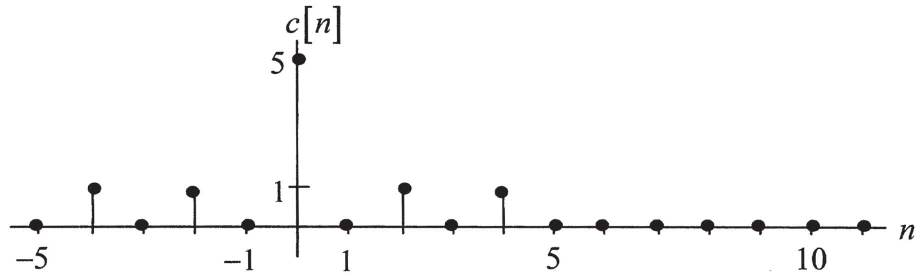


This new input when convolved with $h[n]$ will give the circular convolution found in (c)

We merely extend $x[n]$ as a periodic signal with period 6 samples.

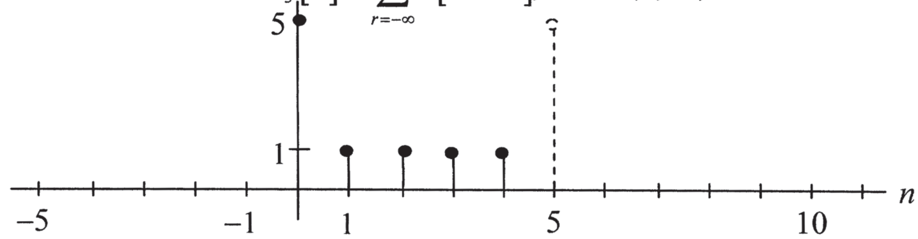
(f) In general $x_1[n]$ is constructed by extending $x[n]$ periodically for $n = -1, K, -M$.

8.29. (a)



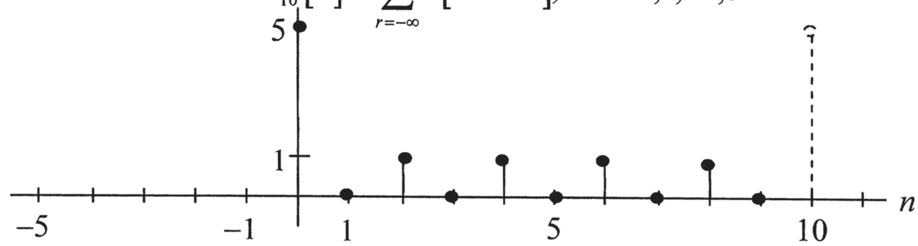
(b)

$$c_5[n] = \sum_{r=-\infty}^{\infty} c[n - r5], \quad n = 0, 1, \ldots, 4$$

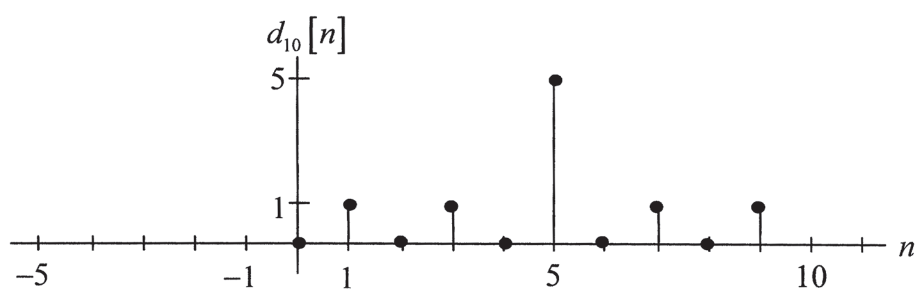


(c)

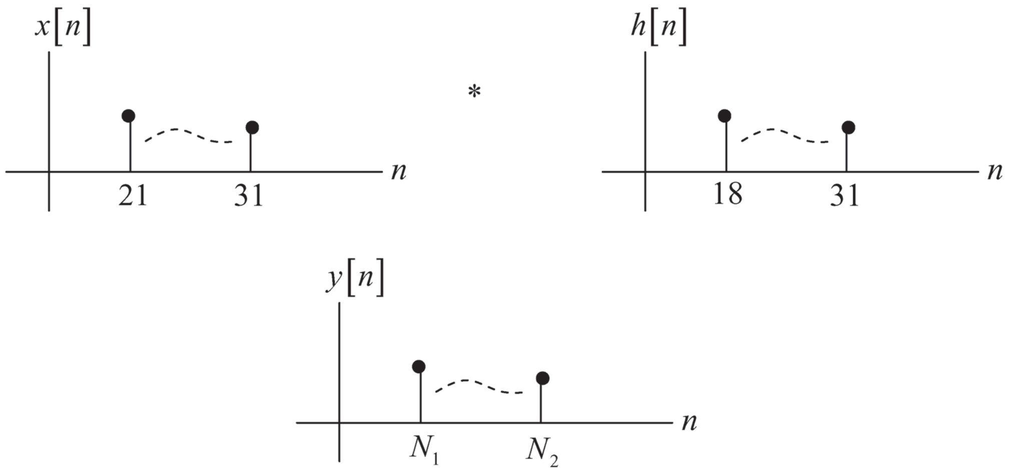
$$c_{10}[n] = \sum_{r=-\infty}^{\infty} c[n - r10], \quad n = 0, 1, \ldots, 9$$



(d) $W_{10}^{5k} \Rightarrow$ rotate right by 5 .



8.30. A.



$$N_1 = 21 + 18 = 39$$
$$N_2 = 31 + 31 = 62.$$

B. The sequence $y_1[n]$ is the 32-point circular convolution of $x_1[n]$ with $h_1[n]$. That is,

$$y_1[n] = \sum_{r=-\infty}^{\infty} y[n + r32]$$
$$= y[n + 32], \quad n = 0, 1, \ldots 31,$$

since $y[n+32]$ is the only one that fits in $0 \le n \le 31$.

C. If we add zeros at the ends too, we can get $y_1[n] = y[n]$ if $N > 62$.

8.31. Given $x[n] = 2\delta[n] + \delta[n-1] - \delta[n-2]$,

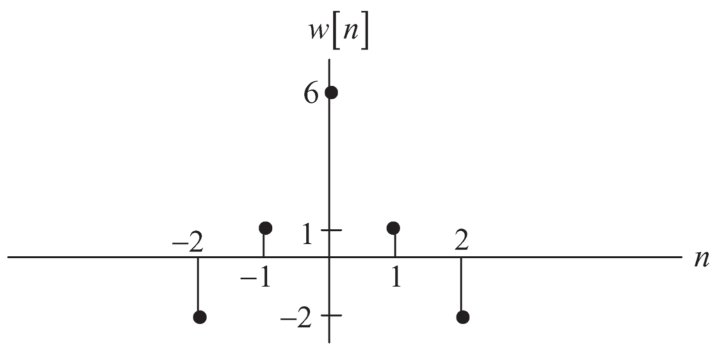
A. $X(e^{j\omega}) = 2 + e^{-j\omega} - e^{-j\omega 2}$.

$Y(e^{j\omega}) = 2 + e^{j\omega} - e^{j\omega 2}$ for $y[n] = x[-n]$.

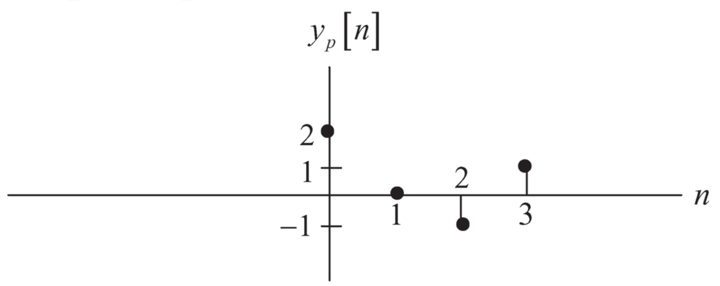
B. If $W(e^{j\omega}) = X(e^{j\omega})Y(e^{j\omega})$, then

$$W(e^{j\omega}) = (2 + e^{-j\omega} - e^{-j\omega 2})(2 + e^{j\omega} - e^{j\omega 2})$$
$$= -2e^{j\omega 2} + e^{j\omega} + 6 + e^{-j\omega} - 2e^{-j\omega 2}.$$

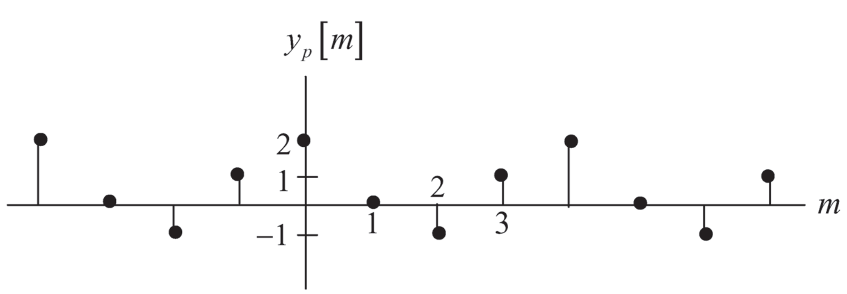
C.

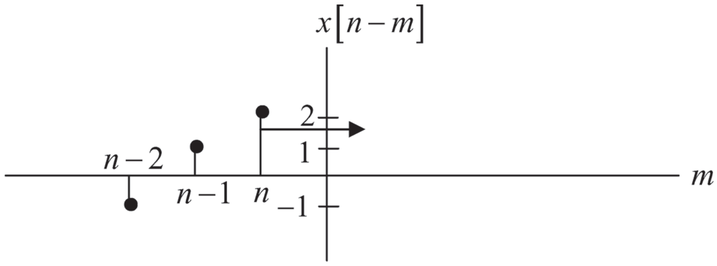


D. Define $y_p[n] = x[((-n))_4]$.

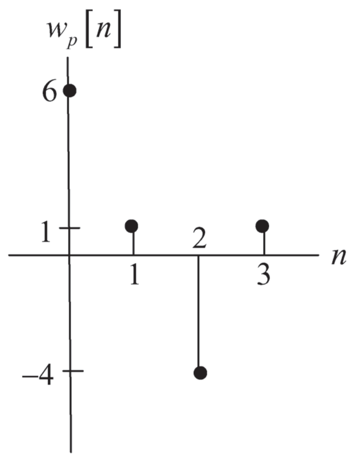


E. To circularly convolve $x[n]$ with $y_p[n]$, consider the construction shown below.





The result, $w_p[n]$ is plotted below.



Note that $w_p[n]$ is periodic; only one period is shown in the plot.

F. Since $x[n]$ has three contiguous non-zero samples, there will be no time-domain aliasing if $N \geq 5$. The circular convolution of $x[n]$ with $x[((-n))_5]$ will be identical to $w[n]$ plotted in C above.

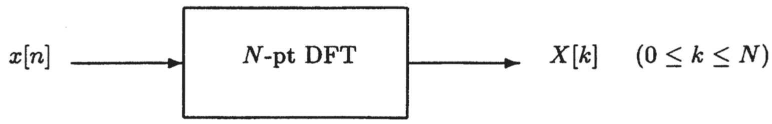
8.32. We have $x[n]$ for $0 \le n \le P$.

We desire to compute $X(z)|_{z=e^{-j(2\pi k/N)}}$ using one N-pt DFT.

(a) Suppose $N > P$ (the DFT size is larger than the data segment). The technique used in this case is often referred to as zero-padding. By appending zeros to a small data block, a larger DFT may be used. Thus the frequency spectra may be more finely sampled. It is a common misconception to believe that zero-padding enhances spectral resolution. The addition of a larger block of data to a larger DFT would enhance this quality.

So, we append $N_z = N - P$ zeros to the end of the sequence as follows:

$$x'[n] = \begin{cases} x[n], & 0 \le n \le (P-1) \\ 0, & P \le n \le N \end{cases}$$



(b) Suppose $N > P$, consider taking a DFT which is smaller than the data block. Of course, some aliasing is expected. Perhaps we could introduce time aliasing to offset the effects.

Consider the N-pt inverse DFT of $X[k]$,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \qquad 0 \le n \le (N-1)$$

Suppose $X[k]$ was obtained as the result of an infinite summation of complex exponents:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{\infty} x[m]e^{-j(2\pi k/N)m}\right) W_N^{-kn}$$

Rearrange to get:

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\left(\frac{1}{N}\sum_{k=0}^{N-1} e^{-j(2\pi/N)(m-n)k}\right)$$

Using the orthogonality relationship of Example 8.1:

$$\begin{aligned} x[n] &= \sum_{m=-\infty}^{\infty} x[m] \sum_{r=-\infty}^{\infty} \delta[m-n+rN] \\ x[n] &= \sum_{r=-\infty}^{\infty} x[n-rN] \end{aligned}$$

So, we should alias $x[n]$ as above. Then we take the N-pt DFT to get $X[k]$.

(b) Using equation (8.171),

$$\sum_{n=0}^{N-1} |X^{c2}[k]|^2 = \sum_{n=0}^{N-1} |X_2[k]|^2.$$

Note that, using equation (8.167):

$$\sum_{k=0}^{2N-1} |X_2[k]|^2 = 2\sum_{k=0}^{N-1} |X[k]|^2 - |X[0]|^2,$$

and, using equation (8.166):

$$\sum_{n=0}^{2N-1} |x_2[n]|^2 = 2\sum_{n=0}^{N-1} |x[n]|^2.$$

Using the DFT properties:

$$\sum_{n=0}^{2N-1} |x_2[n]|^2 = \frac{1}{2N}\sum_{k=0}^{2N-1} |X_2[k]|^2.$$

We thus conclude:

$$\frac{1}{2N}(2\sum_{k=0}^{N-1} |X[k]|^2 - |X[0]|^2) = 2\sum_{n=0}^{N-1} |x[n]|^2.$$