Student name: Student number:

MIDTERM EXAMINATION
ECE431H1F, Digital Signal Processing

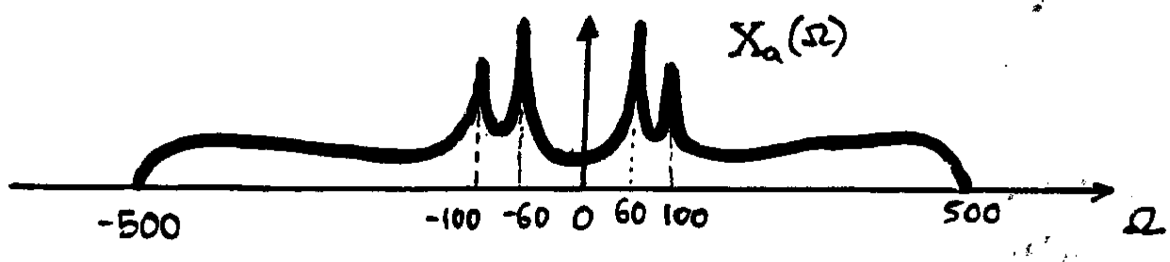
October 20, 2011
Examiner: D. Hatzinakos

Time: 6:15-7:45 pm, Room HA316

This is a closed book exam. You may use calculators and any of the provided information.

Exam questions

PROBLEM It is desired to use a N-DFT algorithm to compute samples of the spectrum of a continuous time signal $x_a(t)$. The spectrum $X_a(\Omega)$ of $x_a(t)$ is given in the next figure.



a) What is the minimum sampling rate $\frac{1}{T}$ required for the discritization (digitization) of $x_a(t)$? What is the minimum number of spectral samples (i.e., minimum N) required to preserve the important characteristics of $x_a(t)$? What is the required minimum length of $x_a(t)$ in seconds? Justify your answers. (3 points)

To obtain sufficient spectral resolution for the detection of the two peaks in the frequency domain spacing between samples of the N-DFT should correspond to $\Delta\Omega \leq 40$ rad/sec

Thus,

Minimum sampling required (Nyquist theorem) : $\Omega_s = 1000$ rad/sec. because signal is bandlimited.

Minimum number of spectral samples: $\frac{\Omega_s}{\Delta\Omega} = \frac{100\cancel{0}}{4\cancel{0}} = 25$ $(N \geq 25)$

Minimum length of $x_a(t)$ required is: $25 \cdot T = 25 \cdot \frac{2\pi}{\Omega_s} = 50\pi$ msec

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b) What is the relationship between the N-DFT (denoted by $X(k)$) obtained in part (a) and $X_a(\Omega)$? Suppose we compute a $2N - DFT$ (denoted by $X'(k)$) by appending N zeros to the signal $x(n) = x_a(t = nT)$, $n = 0, 1, \ldots, N-1$. What is the relation between $X(k)$ and $X'(k)$? Provide the corresponding relations explicitly. (3 points)

Let $W(\Omega)$ be the Fourier transform of a rectangular window of length 50π msec.
Also, let $Q(\Omega) = X(\Omega) * W(\Omega)$ where $*$: convolution

- Then, $X(k) = \frac{1}{T}\sum_{\ell} Q\left(\Omega + \frac{2\pi\ell}{T}\right)\Big|_{\Omega = \frac{2\pi k}{NT}}$, $k = 0, 1, \ldots, N-1$

- $X(k) = X'(2k)$, $k = 0, 1, \ldots, N-1$

c) An analysis of the spectrum of X(k) indicates two spectral peaks at ω=40 T rad/sample and ω= 120 T rad/sample while X'(k) seems to show that the peaks have moved over to ω=50 T rad/sample and ω=110 T rad/sample. What has happened? Be explicit. (3 points)

$X'(k)$ provides a more dense representation of the original spectrum and therefore better "visual resolution".

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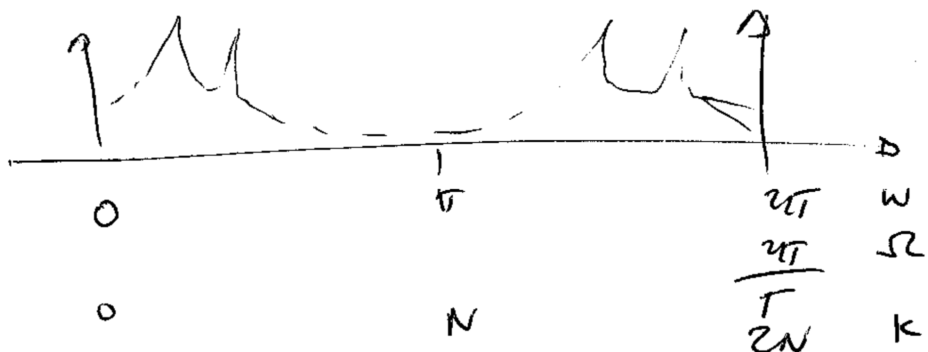
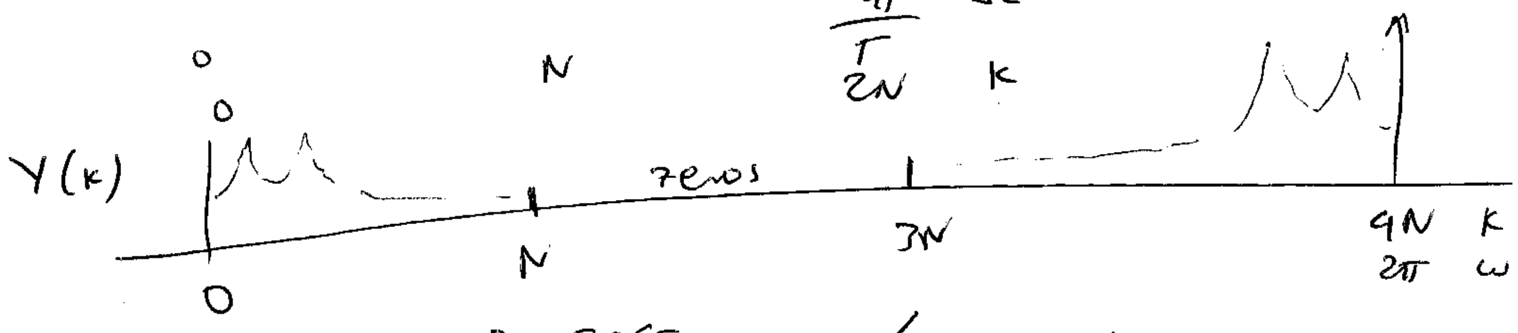
d) As we append more zeros and continue to compute larger DFTs, the spectral peaks tend to concentrate around $\omega=60\ T$ rad/sample and $\omega=100\ T$ rad/sample. Can we conclude that the analog spectrum $Xa(\Omega)$ has indeed two peaks at $\Omega=60$ rad/sec and $\Omega=100$ rad/sec? Justify your answer. (3 points)

If there is aliasing introduced in the transformation of $x_a(t)$ from analog to digital then we can can conclude that $\Omega = 60$ r/s, $\Omega = 100$ r/s are the true peaks. (which is always the case in practice since we consider finite length time signals)

e) Form a sequence $Y(k)$ as follows: $Y(k) = [X'(k)]^2$, $k = 0, 1, \ldots, N$, $Y(k) = 0$, $k = N + 1, \ldots 3N$, $Y(k) = [X'(k-2N)]^2$, $k = 3N+1, \ldots, 4N-1$ and take the 4N-IDFT of $Y(k)$ to obtain the sequence $y(n)$, $n = 0, 1, \ldots, 4N-1$. What is the relation between $y(n)$ and $x(n)$? (4 points)

$X'(k)$ is a 2N-point DFT





$$Y(k) = X'(k)\cdot X'(k) \xrightarrow{2N\text{-IDFT}} y(n) = x'(n) \circledast_{2N} x'(n) = x(n) * x(n)$$

By zero padding $X'(k)$ with 2N zeros, we interpolate 2N zeros in time domain

So if $g(n) = x(n) * x(n)$

$y(2n) = 2g(n)$ $n = 0, 1, \ldots 2N-1$

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f) You are given an N-DFT subroutine that implements $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}$, $k = 0,\ldots,N-1$. Can you demonstrate a clever way to evaluate equally spaced samples of $X(\omega)$ at $\omega = \left(\frac{2\pi k}{N} + \frac{\pi}{N}\right)$, $k = 0,\ldots,N-1$ by using the given subroutine? (4 points)

We want to calculate

$$X(\omega)\Big|_{\omega=\frac{2\pi k}{N}+\frac{\pi}{N}} = \sum_{n=0}^{N-1} x(n)\, e^{-j\omega n}\Big|_{\omega=\frac{2\pi k}{N}+\frac{\pi}{N}} =$$

$$= \sum_{n=0}^{N-1} x(n)\cdot e^{-j\frac{2\pi k n}{N}}\, e^{-j\frac{\pi n}{N}} =$$

$$= \underbrace{\sum_{n=0}^{N-1}\left(x(n)\cdot e^{-j\frac{\pi n}{N}}\right) e^{-j\frac{2\pi k n}{N}}}_{N\text{-DFT of } x(n)\cdot e^{-j\frac{\pi n}{N}}}$$

So