FINAL EXAMINATION
ECE431H1F, Digital Signal Processing

December 16, 2011
Examiner: D. Hatzinakos

Time: 9:30 am-12:00 noon, Room WB119

1. This is a closed book exam (Type A)
2. Non-programmable calculators are allowed.
3. Please solve all five problems. All problems are equally weighted,
4. All answers must be written in the examination booklet. Do not write any answers in this problem handout.

PROBLEM 1 (10 points)
Compute and draw the N-DFT of the length-N sequence

$$x(n) = \cos(2\pi r n / N), \qquad 0 \le n \le N-1$$

where r is an integer in the range $0 \le n \le N-1$.

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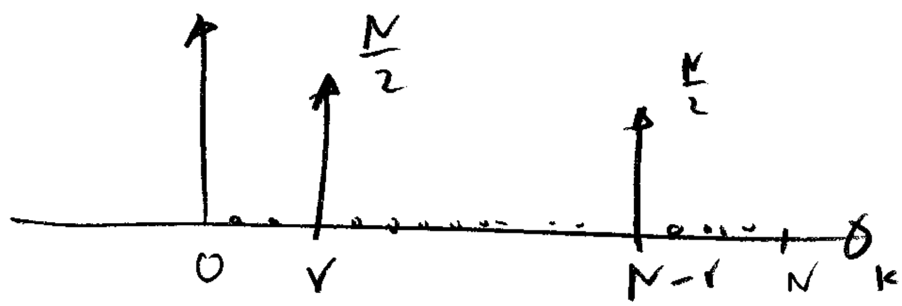
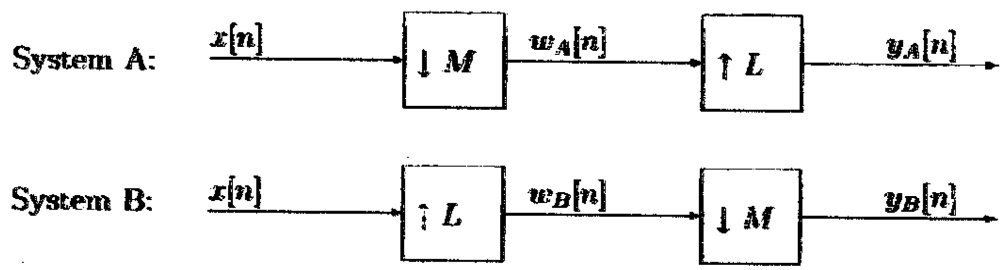
$$X(k) = \sum_{n=0}^{N-1} \cos\left(\frac{2\pi n r}{N}\right) e^{-j\frac{2\pi k n}{N}} = \sum_{n=0}^{N-1} \frac{1}{2}\left(e^{j\frac{2\pi r n}{N}} + e^{-j\frac{2\pi r n}{N}}\right) e^{-j\frac{2\pi k n}{N}}$$

$$= \frac{1}{2} \sum_{n=0}^{N-1} \left(e^{-j\frac{2\pi n (k-r)}{N}} + e^{-j\frac{2\pi n (r+k)}{N}} \right) =$$

$$= \frac{N}{2}\,\delta(k-r+N\ell) + \frac{N}{2}\,\delta(k+r+N\ell) - \ldots$$



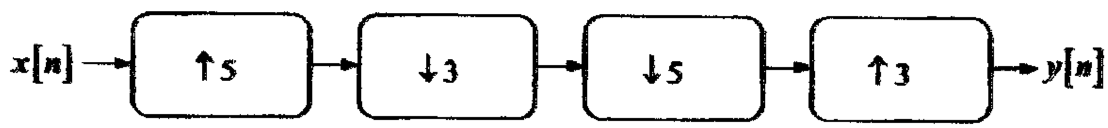
PROBLEM 2 (10 points)
Consider the two systems in the next figure:



(a) For M=2, L=3 and any arbitrary *x[n]* will $y_A[n] = y_B[n]$? If your answer is yes, justify your answer. If your answer is no , clearly explain or give a counterexample

(b) Verify that $y_A[n] = y_B[n]$ for any x[n], if and only if M and L are relatively prime integers. (Hint: Use a time domain proof)

(c) For the system shown below find an expression for *y[n]* in terms of *x[n]*. Simplify the expression as much as possible.



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Solution from Spring05 PS3

(a) The following equations describe the stages of System A:

$$w_A[n] = x[2n]$$

$$y_A[n] = \begin{cases} w_A[\frac{n}{3}] & \text{if } \frac{n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

The following equations describe the stages of System B:

$$w_B[n] = \begin{cases} x[\frac{n}{3}] & \text{if } \frac{n}{3} \text{ is an integer.} \\ 0 & \text{otherwise.} \end{cases}$$

$$y_B[n] = w_B[2n]$$

Therefore,

$$y_A[n] = \begin{cases} x[\frac{2n}{3}] & \text{if } \frac{n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$y_B[n] = \begin{cases} x[\frac{2n}{3}] & \text{if } \frac{2n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

Because for all integer values of n for which $\frac{n}{3}$ is an integer, $\frac{2n}{3}$ is also an integer and vice-versa. the systems are equivalent.

(b) More generally, the systems can be described by the following equations:

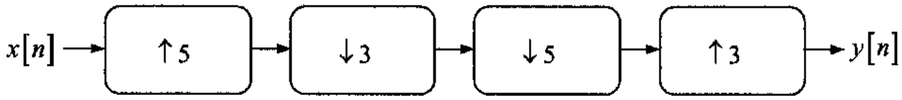
$$y_A[n] = \begin{cases} x[\frac{Mn}{L}] & \text{if } \frac{n}{L} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

$$y_B[n] = \begin{cases} x[\frac{Mn}{L}] & \text{if } \frac{Mn}{L} \text{ is an integer.} \\ 0 & \text{otherwise.} \end{cases}$$

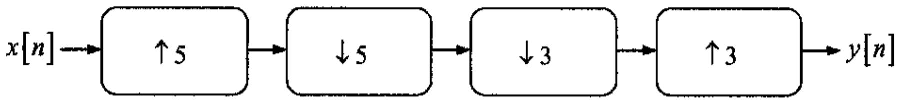
Therefore the two systems are equivalent if for all integer values of n where $\frac{Mn}{L}$ is an integer, $\frac{n}{L}$ is also an integer, and if for all integer values of n where $\frac{n}{L}$ is an integer, $\frac{Mn}{L}$ is also an integer. Since we are guaranteed that for each n which gives integer values of $\frac{n}{L}$, $\frac{Mn}{L}$ must also be an integer (since we're only considering integer M and L), we need only to show that every time $\frac{Mn}{L}$ is an integer, $\frac{n}{L}$ is an integer in order to have an equivalence between the two systems.

For arbitrary integer n, $\frac{Mn}{L}$ is an integer if and only if Mn is an integral multiple of L. This only occurs whenever Mn contains all of L's prime factors. Likewise, $\frac{n}{L}$ is an integer if and only if n contains all of L's prime factors. It is therefore true that in order for the systems to be equivalent, Mn containing all of L's prime factors must imply that n contains all of L's prime factors. This is guaranteed to be true if M and L share no prime factors in common besides 1. (This condition will ensure that any prime factors which Mn has in common with L, besides 1, must have come exclusively from n.) Therefore, the two systems are equivalent if the greatest common factor of M and L is 1 (M and L are co-prime).

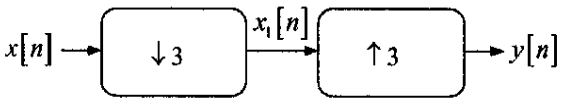
4.33.



Since 3 and 5 are relatively prime, the order of the two operations in the center can be interchanged. This gives



Expanding by 5 and immediately compressing by 5 produces no net effect. We have



Compressing by 3 produces

$$x_1[n] = x[3n].$$

Expanding by 3 now gives

$$y[n] = \begin{cases} x_1[n/3], & n = 3k, \quad k \text{ any integer} \\ 0, & \text{otherwise.} \end{cases}$$

That is,

$$y[n] = \begin{cases} x[n], & n = 3k, \quad k \text{ any integer} \\ 0, & \text{otherwise.} \end{cases}$$

PROBLEM 3 (10 points)

A causal LTI system has system function given by the following expression:

$$H(z) = \frac{1}{1-z^{-1}} + \frac{1-z^{-1}}{1-z^{-1}+0.8z^{-2}}.$$

(a) Is this system stable? Explain briefly.
(b) Draw the signal flow graph of a parallel form implementation of this system.
(c) Draw the signal flow graph of a cascade form implementation of this system as a cascade of 1st order system and a 2nd order system. Use a transposed direct form II implementation for the 2nd order system.

a) System has a pole at $z=1$. Not stable

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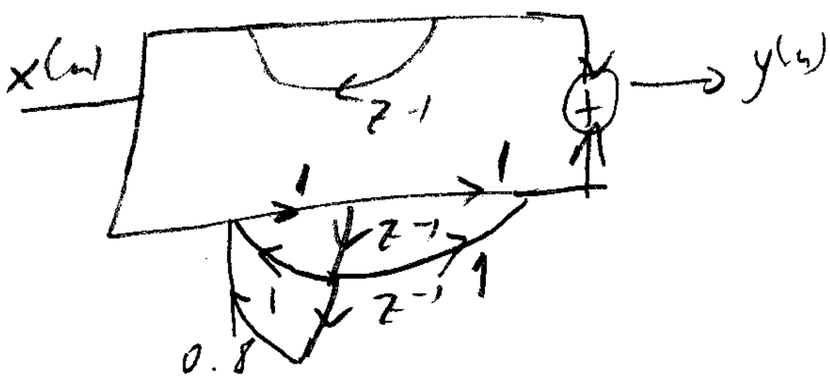
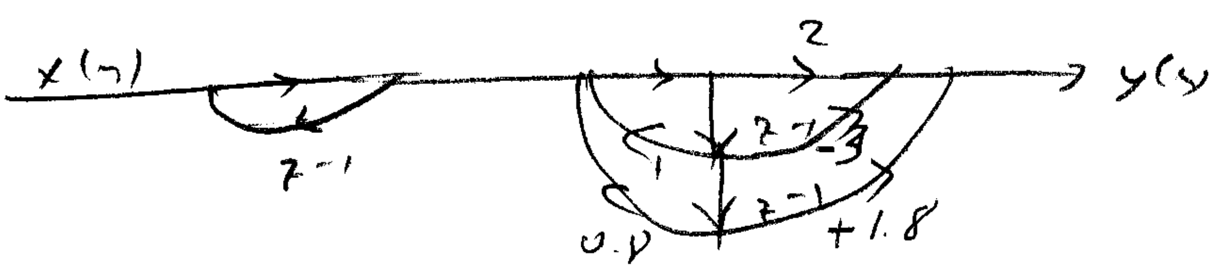
b)



c) $H(z) = \frac{(1-z^{-1}+0.8z^{-2})+(1-2z^{-1}+z^{-2})}{(1-z^{-1})(1-z^{-1}+0.8z^{-2})} = \frac{1}{(1-z^{-1})} \cdot \frac{(2-3z^{-1}+1.8z^{-2})}{(1-z^{-1}+0.8z^{-2})}$



PROBLEM 4 (10 points)

(a) You are given a number of 3-point DFT chips. Show explicitly how you should interconnect four (4) such chips to compute a 12-point DFT (include diagram and expressions). What is the number of multiplications and additions required? What are the savings in computations compared to direct computation of 12-point DFT?

(b) How can you use the 12-FFT you designed in part (a) to compute an inverse 12-DFT algorithm?

a)

$$X(k) = \underbrace{\sum_{m=0}^{2} x(4m) W_3^{mk}}_{3\text{-DFT}} + W_{12}^{k} \underbrace{\sum_{m=0}^{2} x(4m+1) W_3^{mk}}_{3\text{DFT}} + W_{12}^{2k} \underbrace{\sum_{m=0}^{2} x(4m+2) W_3^{mk}}_{3\text{-DFT}}$$

$$+ W_{12}^{3k} \underbrace{\sum x(4m+3) W_3^{mk}}_{3\text{DFT}!} ; \quad k = 0, 1, \ldots, 11$$

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and so on

b) $$x(n) = \frac{1}{12} \sum_{k=0}^{12} X(k) W_{12}^{-nk}$$

→ Interchange input – output

→ Replace each W^{mk}, W_{12}^{nk} with W_3^{-mk}, W_{12}^{-2k} ———

→ Multiply each output by $\frac{1}{12}$

PROBLEM 5 (10 points)

(a) a sinusoidal signal with peak to-peak amplitude of 10 Volts is digitized with a 12 bit ADC. Assuming linear quantization, determine

- the quantization step size
- the quantization noise power
- the theoretical maximum SQNR

(b) Realize the following transfer function of a causal stable filter

$$H(z) = \frac{(z+0.3)}{(z-0.4)} = \frac{1+0.3z^{-1}}{1-0.4z^{-1}}$$

in direct form I implementation. Assuming fixed point implementation and making all necessary assumptions, show the corresponding noise model for the computation of the product round-off noise at the output of this realization. Compute explicitly the output round-off quantization noise variance.

a) $\Delta = \frac{10}{2^{12}}$, $\sigma_n^2 = \frac{\Delta^2}{12}$

$SQNR = 6.02\,b + 10.8$ dB

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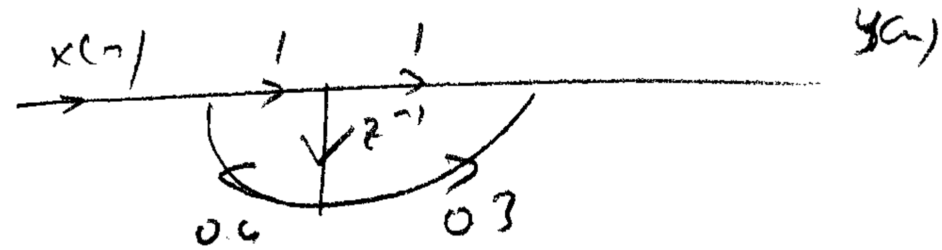
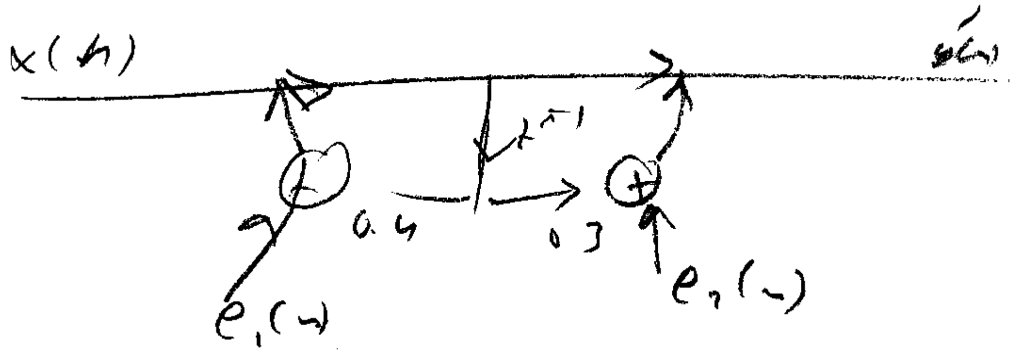
b)



fp fixed p.t



$$y(n) = \underbrace{x(n) * h(n)}_{y(n)} + \underbrace{e_2(n) + e_1(n) * h(n)}_{n(n)} \Rightarrow$$

$$\Rightarrow \sigma_n^2 = \sigma_e^2 + \sigma_e^2 \sum_n h^2(n)$$

$h(n) = 0.4^n u(n) + 0.3 \cdot 0.4^{(n-1)} u(n-1)$