The Kalman Filter as the Optimal Linear Minimum Mean-Squared Error Multiuser CDMA Detector

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***Abstract*—In this paper, it is shown that a first-order linear state–space model applies to the asynchronous code-division multiple-access (CDMA) channel, and thus the Kalman filter produces symbol estimates with the minimum mean-squared error (MMSE) among all linear filters, in long- or short-code systems for a given detection delay. This result may be used as a benchmark against which to compare the performance of other linear detectors in asynchronous channels. It also reveals that a time-varying recursive filter with a fixed and finite complexity implements the fixed-lag linear MMSE (LMMSE) detector, which hitherto has been assumed to require a processing window (and hence complexity) that grows with time.**

***Index Terms*—Code division multiple access (CDMA), Kalman filtering, linear minimum mean-squared error (MMSE) detection, multiuser communications.**

I. Introduction

A CLASS of code-division multiple-access (CDMA) receivers known as linear minimum mean-squared error (MMSE) detectors [1, Ch. 6] has been discussed in recent years. The MMSE detector is commonly assumed to be a tapped-delay line (one for each desired user), meaning that the decision statistic for the kth user in the ith symbol interval will be

$$\hat{d}_k(i) = \boldsymbol{f}_{k,0}^H(i)\boldsymbol{r}(i) \tag{1}$$

where $\boldsymbol{f}_{k,0}(i)$ is the MMSE filter tap-weight vector, while $\boldsymbol{r}(i)$ is the received discrete-time signal vector of the same length as $\boldsymbol{f}_{k,0}(i)$.

With this tapped-delay line (TDL) structure, and the reasonable assumption that the symbols transmitted are independent, i.e., $E[\boldsymbol{d}(i)\boldsymbol{d}^H(i)] = \boldsymbol{I}$, where $\boldsymbol{d}(i)$ is the vector of symbols which contribute to $\boldsymbol{r}(i)$ and $\boldsymbol{I}$ is the identity matrix, it may be shown that

$$\boldsymbol{f}_{k,0}(i) = [\boldsymbol{A}(i)\boldsymbol{A}^H(i) + \sigma^2\boldsymbol{I}]^{-1}\boldsymbol{a}_k(i). \tag{2}$$

The signal model used here is $\boldsymbol{r}(i) = \boldsymbol{A}(i)\boldsymbol{d}(i) + \boldsymbol{n}(i)$, where $\boldsymbol{A}(i)$ is the channel matrix, $\boldsymbol{a}_k(i)$ is the signal vector received from the desired symbol $d_k(i)$, and $\boldsymbol{n}(i)$ is an additive white Gaussian noise (AWGN) vector with covariance matrix

$$E[\boldsymbol{n}(i)\boldsymbol{n}^H(i)] = \sigma^2\boldsymbol{I}.$$

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While $\boldsymbol{f}_{k,0}(i)$ is the MMSE tap-weight vector for a given tap length, it does not produce the lowest MMSE of all linear filters. The optimal linear MMSE (LMMSE) detector spans all symbols transmitted by all users, i.e., it is equivalent to a TDL with an infinite number of taps. For a finite detection delay, the TDL MMSE detector still needs to span the infinite past.

The Kalman filter, on the other hand, being a first-order recursive filter, naturally processes *all* information collected up to a given point in time. It produces state estimates that are optimal in the LMMSE sense, if the system is accurately modeled by a set of first-order state–space equations. It is shown in the next section that a state–space model can indeed be formulated for the multiuser CDMA signal, with the unknown transmitted symbol vector $\boldsymbol{d}(i)$ as the state vector. Thus the Kalman filter must necessarily lead to the lowest possible mean-square error (MSE) among all linear filters for a given detection delay.

The application of the Kalman filtering concept to CDMA detection was also described in [2], [3], but the state–space model used there was less flexible in that the detection delay was always zero, i.e., detection of any symbol had to be made immediately after it was received. In the present paper, we allow for an arbitrary detection delay, with longer delays leading to better performance. The model in [2] and [3] is therefore a subset of the one to be described here. In addition, in this paper, we use a symbol-rate vector observation equation, whereas in our previous work a chip-rate scalar observation was used.

Our results demonstrate that it is possible in simulations to find the performance of the fixed-lag linear MMSE detector in an asynchronous system. With the semianalytical bit-error rate (BER) expressions for AWGN channels that will be derived later, Monte Carlo simulations of the Kalman filter to obtain the BER of the LMMSE detector are not necessary, thereby shortening simulation times considerably. We therefore envision that the results presented here can be used for benchmarking the performance of linear detectors, such as multistage linear interference cancellers [4], [5], whose aim is to approach MMSE performance in asynchronous channels.

II. A State-Space Model of the Multiuser CDMA System

The baseband received signal in a CDMA system with K active users is normally sampled at M $(M > 1)$ times the chip rate to give the complex sequence $r(n)$, and then processed to yield the symbol decisions for some or all of the users. For mathematical convenience, the high-rate sequence $r(n)$ may be "vectorized" by stacking consecutive elements in vectors $\boldsymbol{r}(i)$, as shown in Fig. 1, where i denotes the symbol index. $\boldsymbol{r}(i)$ is, therefore, a symbol-rate representation of the received signal.

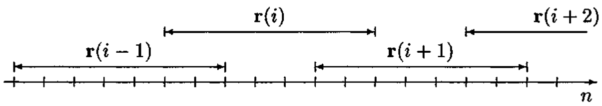

Fig. 1. Converting the high-rate received signal sequence $r(n)$ to an equivalent symbol-rate vector representation $\boldsymbol{r}(i)$. The tick marks represent samples of $r(n)$, and the symbol interval is five times the sampling interval.

The span of $\boldsymbol{r}(i)$ may exceed one symbol interval, for instance when a windowed implementation of the linear MMSE detector is desired in an asynchronous scenario [1]. As long as *all* samples of $r(n)$ belong to at least one vector $\boldsymbol{r}(i)$, no information will be lost, and the new symbol-rate vector representation will carry the same information as the high-rate scalar one.

As explained in [6], whatever the channel delay spread and the length of $\boldsymbol{r}(i)$, it can always be written as

$$\begin{aligned}\boldsymbol{r}(i) &= \boldsymbol{S}(i)\boldsymbol{C}(i)\boldsymbol{d}(i) + \boldsymbol{n}(i)\\ &= \boldsymbol{A}(i)\boldsymbol{d}(i) + \boldsymbol{n}(i)\end{aligned} \tag{3}$$

where $\boldsymbol{S}(i)$ is a matrix of spreading code vectors, $\boldsymbol{C}(i)$ is a block-diagonal matrix of channel coefficient vectors, $\boldsymbol{n}(i)$ is a zero-mean Gaussian noise vector with covariance matrix $\sigma^2\boldsymbol{I}$, and $\boldsymbol{d}(i)$ is the vector of transmitted symbols from all users which contribute to $\boldsymbol{r}(i)$.

The detection delay P, in symbol intervals, is defined to be the time between the last sample of the signal due to the desired symbol $d_k(i)$ arriving, and the arrival of the last sample of the received signal used in its detection. It is a design parameter that is not necessarily an integer. To illustrate this definition, three examples are provided in Fig. 2, where we have also defined Q to be the time (in terms of symbols) between the first samples of $\boldsymbol{r}(i)$ and the signal due to $d_k(i)$ arriving.

Basically, with the received signal up to and including $\boldsymbol{r}(i)$, we wish to demodulate the symbol $d_k(i)$, which lies in the subvector $\bar{\boldsymbol{d}}(i - \lfloor P \rfloor)$ or $\bar{\boldsymbol{d}}_b(i - \lceil P \rceil)$ when $\boldsymbol{d}(i)$ is partitioned into K-element subvectors as follows:

$$\boldsymbol{d}(i) \stackrel{\text{def}}{=} \left[\bar{\boldsymbol{d}}_b^T(i - \lceil P \rceil), \bar{\boldsymbol{d}}^T(i - \lfloor P \rfloor), \ldots, \bar{\boldsymbol{d}}^T(i-1), \bar{\boldsymbol{d}}^T(i)\right]^T. \tag{4}$$

Because the dimension N_d of $\boldsymbol{d}(i)$ is not necessarily a multiple of K, $\bar{\boldsymbol{d}}_b(i - \lceil P \rceil)$ is needed to account for the bottom $N_d - \lfloor N_d/K \rfloor K$ elements of $\bar{\boldsymbol{d}}(i - \lceil P \rceil)$.

From (4), it is clear that there will be K new symbols contributing to the received signal vector with each increment in i. In fact, $\boldsymbol{d}(i)$ satisfies the following first-order (non-Gaussian) Markov transition model:

$$\begin{aligned}\boldsymbol{d}(i) &= \begin{bmatrix}\boldsymbol{0} & \boldsymbol{I}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\boldsymbol{d}(i-1) + \begin{bmatrix}\boldsymbol{0}\\ \bar{\boldsymbol{d}}(i)\end{bmatrix}\\ &= \boldsymbol{\Phi}_d\boldsymbol{d}(i-1) + \boldsymbol{w}(i)\end{aligned} \tag{5}$$

where $\boldsymbol{w}(i)$ is a random vector with the known covariance matrix

$$E[\boldsymbol{w}(i)\boldsymbol{w}^H(i)] = \boldsymbol{Q}_w = \begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix}$$

if phase-shift keying (PSK) with unit symbol energy is used. Equation (5) provides the state transition equation needed in the Kalman filter, and $\boldsymbol{d}(i)$ is the "state vector" to be estimated by the Kalman filter.

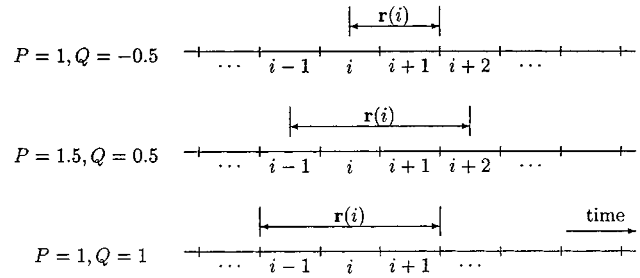

Fig. 2. Illustrations of how different P's and Q's affect the observation window. The tick marks represent symbol boundaries for user k, the desired user.

III. Applying the Kalman Filter

A. Small Detection Delays

With the TDL detector structure, it is necessary for good performance to use observation windows for $\boldsymbol{r}(i)$ that exceed one symbol interval. With the Kalman filter, the choice of window size depends on the detection delay P. The one basic requirement is that all samples of $r(n)$ are used, so that the minimum window span is one symbol interval. Windows spanning more than one symbol interval are not required unless $P \geq 1$, in which case the minimum length of $\boldsymbol{r}(i)$ is $PMN + 1$ samples.[1]

With nonoverlapping windows, the measurement noise vector $\boldsymbol{n}(i)$ is white, and

$$E[\boldsymbol{n}(i)\boldsymbol{n}^H(j)] = \sigma^2\boldsymbol{I}\delta_{ij}$$

where

$$\delta_{ij} = \begin{cases}1, & i = j\\ 0, & i \neq j.\end{cases}$$

In this case, the multiuser CDMA state–space model corresponds to the basic one used in Kalman filtering, in which the measurement noise and state noise are both white, and mutually uncorrelated.

Therefore, it is straightforward to implement the Kalman filter with knowledge of the channel matrix $\boldsymbol{A}(i)$, and the AWGN variance σ^2. In other words, the amount of information required is identical to that in the MMSE detector. The "recursive filter" form of the Kalman filter involves the following steps in each symbol interval i:

$$\boldsymbol{P}_i^- = \boldsymbol{\Phi}_d\boldsymbol{P}_{i-1}^+\boldsymbol{\Phi}_d^H + \boldsymbol{Q}_w \tag{6}$$

$$\boldsymbol{K}_i = \boldsymbol{P}_i^-\boldsymbol{A}^H(i)(\boldsymbol{A}(i)\boldsymbol{P}_i^-\boldsymbol{A}^H(i) + \sigma^2\boldsymbol{I})^{-1} \tag{7}$$

$$\boldsymbol{P}_i^+ = (\boldsymbol{I} - \boldsymbol{K}_i\boldsymbol{A}(i))\boldsymbol{P}_i^- \tag{8}$$

$$\hat{\boldsymbol{d}}(i) = (\boldsymbol{I} - \boldsymbol{K}_i\boldsymbol{A}(i))\boldsymbol{\Phi}_d\hat{\boldsymbol{d}}(i-1) + \boldsymbol{K}_i\boldsymbol{r}(i). \tag{9}$$

The vector $\hat{\boldsymbol{d}}(i)$ is the soft estimate of $\boldsymbol{d}(i)$ from which we extract the soft decision for $d_k(i)$ (or any other symbol within $\boldsymbol{d}(i)$, for that matter). In Kalman filter terminology, it is the "filtered state estimate" and is more correctly denoted by $\hat{\boldsymbol{d}}(i \mid i)$—the LMMSE estimate of $\boldsymbol{d}(i)$ given observations up to time i. But since in this paper we do not encounter the "predicted" and "smoothed" estimates ($\hat{\boldsymbol{d}}(i \mid i-1)$ and $\hat{\boldsymbol{d}}(i \mid i+1)$, respectively), we will write $\hat{\boldsymbol{d}}(i)$ in place of $\hat{\boldsymbol{d}}(i \mid i)$ for convenience.

[1] PMN samples give the P symbols' delay, and the one extra sample is the last one received from $d_k(i)$.

The main difference between the Kalman filter and the TDL detector is brought out by (9), which shows that the Kalman filter is a recursive (or feedback) filter, unlike the TDL.

B. Longer Detection Delays

When $P \geq 1$, the observation windows for $\boldsymbol{r}(i)$ will overlap in time, i.e., $\boldsymbol{r}(i)$ and $\boldsymbol{r}(i-j)$ share some elements in common when $j \neq 0$, and the noise vector $\boldsymbol{n}(i)$ is no longer uncorrelated over time. Because of the noise coloring, the Kalman filter as described by (6)–(9) is no longer the LMMSE filter. To derive the LMMSE detector in this scenario, we can do one of two things:

1) reformulate the state–space model so that white measurement noise is present; or
2) use nonoverlapping windows but allow for arbitrary delays through the use of the fixed-lag Kalman smoother, which gives MMSE state estimates at time i given observations up to some time in the future.

As these two approaches lead to identical algorithms, we will describe only the method using the fixed-lag Kalman smoother, and refer the interested reader to [7] for details of the first technique.

The fixed-lag Kalman smoother [8], [9] derives the LMMSE estimate of the state $\boldsymbol{x}(i)$ given observation vectors (in which the additive noise term $\boldsymbol{n}(i)$ is temporally white) up to time $i+L$, $L>0$. A detection delay of PMN samples can therefore be obtained using MN-sample vectors $\bar{\boldsymbol{r}}(i)$[2] and then forming the LMMSE *smoothed* estimates $\hat{d}_k(i \,|\, \bar{\boldsymbol{r}}(i+\lfloor P \rfloor), \ldots, \bar{\boldsymbol{r}}(0))$ with the fixed-lag Kalman smoother.

We recall from Section II that $d_k(i)$ is in either $\bar{\boldsymbol{d}}(i-\lfloor P \rfloor)$ or $\bar{\boldsymbol{d}}(i-\lceil P \rceil)$. Now from (3) and (4), when $\boldsymbol{r}(i)$ spans one symbol interval, we can write

$$\bar{\boldsymbol{r}}(i) = \bar{\boldsymbol{A}}(i-\lfloor P \rfloor)\begin{bmatrix} \bar{\boldsymbol{d}}(i-\lceil P \rceil) \\ \bar{\boldsymbol{d}}(i-\lfloor P \rfloor) \end{bmatrix} + \bar{\boldsymbol{n}}(i) \tag{10}$$

where $\bar{\boldsymbol{n}}(i)$ is the length-MN noise vector affecting $\bar{\boldsymbol{r}}(i)$, and $\bar{\boldsymbol{A}}(i-\lfloor P \rfloor)$ is an $MN \times 2K$ channel matrix.[3]

It follows that

$$\bar{\boldsymbol{r}}(i+\lfloor P \rfloor) = [\boldsymbol{0}, \ldots, \boldsymbol{0}, \bar{\boldsymbol{A}}(i)]\boldsymbol{d}(i) + \boldsymbol{v}(i) \tag{11}$$

where $\boldsymbol{d}(i)$ is the long vector defined in (4), and $\boldsymbol{v}(i) = \bar{\boldsymbol{n}}(i+\lfloor P \rfloor)$. We have basically extended the state vector so that it includes the symbol of interest, and padded zeros to the observation matrix in order to keep the observation equation valid. For convenience, (11) may be rewritten as

$$\boldsymbol{\zeta}(i) = \boldsymbol{A}_\zeta(i)\boldsymbol{d}(i) + \boldsymbol{v}(i) \tag{12}$$

where $\boldsymbol{A}_\zeta(i) = [\boldsymbol{0}, \bar{\boldsymbol{A}}(i)]$ is an $MN \times (\lfloor P \rfloor + 1)K$ matrix, with zeros in all but its last $2K$ columns.

With (12) as the observation equation, and (5) as the state-transition equation, the Kalman filter updates for the arbitrary delay scenario are easily derived. We note also that a careful study of the sparse nature of the matrices involved, along the lines of the method outlined in [8, Sec. 7.3], leads to a substantial reduction in complexity (as shown in [7]), compared to a direct or brute-force implementation of the Kalman filter updates.

[2]This contains the desired symbol $d_k(i)$ and satisfies the white-noise requirement.

[3]Its first few columns will be zero if P is not an integer.

IV. Analysis

A. Steady-State Solution in the Time-Invariant Case

From Kalman filtering theory [9], [8], we have the following result regarding the asymptotic stability of the Kalman filter.

Theorem 1: For the stationary time-invariant state–space model

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{\Phi}\boldsymbol{x}(k) + \boldsymbol{\Psi}\boldsymbol{u}(k) + \boldsymbol{\Gamma}\boldsymbol{w}(k) \\ \boldsymbol{z}(k+1) &= \boldsymbol{H}x(k+1) + \boldsymbol{v}(k+1) \end{aligned}$$

where

$$\begin{aligned} E[\boldsymbol{w}(j)\boldsymbol{v}(k)] &= \boldsymbol{0} \\ E[\boldsymbol{w}(j)\boldsymbol{w}^H(k)] &= \boldsymbol{Q}\delta_{jk} \end{aligned}$$

and

$$E[\boldsymbol{v}(j)\boldsymbol{v}^H(k)] = \boldsymbol{R}\delta_{jk}$$

if the state transition matrix $\boldsymbol{\Phi}$ is asympotically stable (i.e., all its eigenvalues lie within the unit circle), then given any positive semidefinite symmetric initial condition $\boldsymbol{P}^-(0)$, $\lim_{k\to\infty} \boldsymbol{P}^-(k) = \boldsymbol{P}^-_\infty$, where

$$\boldsymbol{P}^-_\infty = \boldsymbol{\Phi}\boldsymbol{P}^-_\infty[\boldsymbol{I} - \boldsymbol{H}^H(\boldsymbol{H}\boldsymbol{P}^-_\infty\boldsymbol{H}^H + \boldsymbol{R})^{-1}\boldsymbol{H}\boldsymbol{P}^-_\infty]\boldsymbol{\Phi}^H + \boldsymbol{\Gamma}\boldsymbol{Q}\boldsymbol{\Gamma}^H. \tag{13}$$

The Kalman gain $\boldsymbol{K}_k$ will also converge to a time-invariant matrix $\boldsymbol{K}_\infty$, and the Kalman filter will be asymptotically stable, i.e.,

$$|\lambda_p[\boldsymbol{\Phi} - \boldsymbol{K}_\infty\boldsymbol{H}\boldsymbol{\Phi}]| < 1, \qquad \forall p \tag{14}$$

where $\lambda_p(\boldsymbol{X})$ represents the pth eigenvalue of $\boldsymbol{X}$.

The state transition model (5) used in this paper is stable, since $\boldsymbol{\Phi}_d$ has zeros on its diagonal and thus only has zero eigenvalues. Therefore, $\boldsymbol{P}^-_i$ in (6) will converge as $i \to \infty$ to $\boldsymbol{P}^-_\infty$, which satisfies

$$\boldsymbol{P}^-_\infty = \boldsymbol{\Phi}\boldsymbol{P}^-_\infty\{\boldsymbol{I} - \boldsymbol{A}^H(\boldsymbol{A}\boldsymbol{P}^-_\infty\boldsymbol{A}^H + \sigma^2\boldsymbol{I})^{-1}\boldsymbol{A}\boldsymbol{P}^-_\infty\}\boldsymbol{\Phi}^T + \boldsymbol{Q}. \tag{15}$$

From this point on, because the Kalman filter and Kalman smoother equations are identical except for trivial redefinitions of the channel matrix and the input signal vector, it should be understood that if the overlapping window model $(P \geq 1)$ is used, $\boldsymbol{A}$ should be replaced by $\boldsymbol{A}_\zeta$, and $\boldsymbol{r}(i)$ by $\boldsymbol{\zeta}(i)$.

From (7), the steady-state Kalman gain matrix $\boldsymbol{K}_\infty$ can be derived as

$$\boldsymbol{K}_\infty = \boldsymbol{P}^-_\infty\boldsymbol{A}^H(\boldsymbol{A}\boldsymbol{P}^-_\infty\boldsymbol{A}^H + \sigma^2\boldsymbol{I})^{-1}. \tag{16}$$

Through simulation (also see examples given in [9] and other textbooks on optimal filtering) it is found that $\boldsymbol{K}_i$ always converges very quickly, usually in a few symbol intervals. After convergence of the Kalman gain, it is unnecessary to continue to update it in each iteration i. This in practice implies that after updating $\boldsymbol{K}_i$ for a few iterations to obtain $\boldsymbol{K}_\infty$, the Kalman gain need not be updated further and only the state estimate update operation needs to be carried out.

The time-invariance of the Kalman filter gain matrices in slowly fading, short-code CDMA systems implies that in implementation, an adaptive infinite impulse response (IIR) filter may be used [10], if training sequences are available and precise channel information is not.

B. Bit-Error Rate Calculations

The recursive nature of the Kalman filter makes exact BER calculations more complicated than with TDL filters. Nevertheless, it is possible to obtain approximate BER expressions for binary phase-shift keying (BPSK) in an AWGN channel that are accurate to an arbitrary level, depending on the complexity that can be tolerated. A much simpler BER computation using the Gaussian assumption can also be derived.

We first define the matrix $\boldsymbol{G}_i = (\boldsymbol{I} - \boldsymbol{K}_i\boldsymbol{A}(i))\boldsymbol{\Phi}_d$, and simplify (9) to[4]

$$\hat{\boldsymbol{d}}(i) = \boldsymbol{G}_i\hat{\boldsymbol{d}}(i-1) + \boldsymbol{K}_i\boldsymbol{r}(i) \tag{17}$$

$$= \prod_{j=1}^{i}\boldsymbol{G}_j\hat{\boldsymbol{d}}(0) + \sum_{n=0}^{i}\left[\prod_{j=i-n+1}^{i}\boldsymbol{G}_j\right]\boldsymbol{K}_{i-n}\boldsymbol{r}(i-n) \tag{18}$$

where, by definition,

$$\prod_{j=n_1}^{n_2}\boldsymbol{G}_j = \begin{cases}\boldsymbol{G}_{n_2}\boldsymbol{G}_{n_2-1}\ldots\boldsymbol{G}_{n_1}, & \text{if } n_2 \geq n_1\\ \boldsymbol{I}, & \text{if } n_2 < n_1.\end{cases}$$

Given that we have a fully observable[5] and stable[6] system, $|\lambda_{\max}(\boldsymbol{G}_i)| < 1$ for all i. Hence, $\prod_{j=1}^{i}\boldsymbol{G}_j$ must decay to $\boldsymbol{0}$ as $i \to \infty$, and for a sufficiently large N_o

$$\sum_{n=0}^{i}\prod_{j=i-n+1}^{i}\boldsymbol{G}_j \approx \sum_{n=0}^{N_o}\prod_{j=i-n+1}^{i}\boldsymbol{G}_j.$$

With these approximations and the aid of (3), we have for large i

$$\hat{\boldsymbol{d}}(i) \approx \sum_{n=0}^{N_o}\left[\prod_{j=i-n+1}^{i}\boldsymbol{G}_j\right]\boldsymbol{K}_{i-n}[\boldsymbol{A}(i-n)\boldsymbol{d}(i-n)+\boldsymbol{n}(i-n)] \tag{19}$$

$$= \boldsymbol{M}(i)\begin{bmatrix}\boldsymbol{d}(i-N_o)\\ \vdots\\ \boldsymbol{d}(i)\end{bmatrix} + \boldsymbol{N}(i)\begin{bmatrix}\boldsymbol{n}(i-N_o)\\ \vdots\\ \boldsymbol{n}(i)\end{bmatrix} \tag{20}$$

where the $N_d \times N_oN_d$ matrices $\boldsymbol{M}(i)$ and $\boldsymbol{N}(i)$ are composed of $N_d \times N_d$ block matrices

$$\left[\prod_{j=i-n+1}^{i}\boldsymbol{G}_j\right]\boldsymbol{K}_{i-n}\boldsymbol{A}(i-n) \quad \text{and} \quad \left[\prod_{j=i-n+1}^{i}\boldsymbol{G}_j\right]\boldsymbol{K}_{i-n}$$

respectively, for n decreasing from N_o to 0. Note that for the Kalman smoother, we will use (10) to make the substitution

$$\boldsymbol{A}(i-n)\boldsymbol{d}(i-n) \to \bar{\boldsymbol{A}}(i-n)\begin{bmatrix}\bar{\boldsymbol{d}}(i-n-\lceil P\rceil)\\ \bar{\boldsymbol{d}}(i-n-\lfloor P\rfloor)\end{bmatrix}$$

in (19). The obvious substitution $\boldsymbol{A}(i) \to \boldsymbol{A}_\zeta(i)$ is not efficient because some columns of $\boldsymbol{A}_\zeta(i)$ are zero. Including these zero columns will unnecessarily lengthen the $\boldsymbol{d}'(i)$ vector below.

[4]Note that, when $\boldsymbol{r}(i)$ is replaced by $\boldsymbol{\zeta}(i)$, and $\boldsymbol{A}(i)$ by $\boldsymbol{A}_\zeta(i)$, (6)–(9) are valid for the Kalman smoother as well. Therefore, the BER analysis given here applies to all values of detection delay P.

[5]Since $\boldsymbol{A}(i)$ is assumed to have full rank.

[6]$\boldsymbol{\Phi}_d$ has spectral radius smaller than unity.

If we consider the detection of $d_k(i)$ (one of the symbols within $\boldsymbol{d}(i)$), the relevant decision statistic will be

$$\hat{d}_k(i) = \boldsymbol{g}_k^T(i)\boldsymbol{d}'(i) + \boldsymbol{h}_k^T(i)\boldsymbol{n}'(i)$$

where $\boldsymbol{g}_k^T(i)$ and $\boldsymbol{h}_k^T(i)$ are, respectively, the rows of the matrices $\boldsymbol{M}(i)$ and $\boldsymbol{N}(i)$ corresponding to $d_k(i)$, from which it appears that the BER can be easily computed.

However, the length-N_oN_d vector

$$\boldsymbol{d}'(i) = [\boldsymbol{d}^T(i-N_o), \ldots, \boldsymbol{d}^T(i)]^T$$

has statistically dependent elements, since successive vectors $\boldsymbol{d}(i)$ and $\boldsymbol{d}(i-1)$ share $(N_d - K)$ common elements. Nonetheless, because of the orderly manner in which $\boldsymbol{d}(i)$ is derived from $\boldsymbol{d}(i-1)$, it is possible to "compress" $\boldsymbol{d}'(i)$, and correspondingly $\boldsymbol{g}_k(i)$, into the vectors $\boldsymbol{d}''(i)$ and $\acute{\boldsymbol{g}}_k(i)$ so that

$$\acute{\boldsymbol{g}}_k^T(i)\boldsymbol{d}''(i) = \boldsymbol{g}_k^T(i)\boldsymbol{d}'(i)$$

and $\boldsymbol{d}''(i)$, unlike $\boldsymbol{d}'(i)$, contains no repeated elements. Similarly, we may define $\acute{\boldsymbol{h}}_k(i)$ such that

$$\acute{\boldsymbol{h}}_k^T(i)\boldsymbol{n}''(i) = \boldsymbol{h}_k^T(i)\boldsymbol{n}'(i)$$

and $\boldsymbol{n}''(i)$ is an uncorrelated Gaussian noise sequence. Details of this procedure may be found in [7].

It is now possible to evaluate the instantaneous bit-error rate using the well-known formula for linear detectors and BPSK modulation:

$$P_{b,k}(i) = \frac{1}{2^{N_d''-1}}\sum_{\substack{d_k(i)=1\\ \text{all }\boldsymbol{d}''(i)}} Q\left(\frac{\acute{\boldsymbol{g}}_k^T(i)\boldsymbol{d}''(i)}{\sqrt{\sigma^2\acute{\boldsymbol{h}}_k^T(i)\acute{\boldsymbol{h}}_k(i)}}\right). \tag{23}$$

In this expression, $N_d'' = (N_0-1)K+N_d$ is the length of $\boldsymbol{d}''(i)$, and σ^2 is the additive noise variance.

Alternatively, one can use the Gaussian assumption which Poor and Verdú [11] showed to be accurate when applied to MMSE detectors, and thus have

$$P_{b,k}(i) = Q\left(\sqrt{\frac{|\acute{g}_k(i)|^2}{\sigma^2\acute{\boldsymbol{h}}_k^T(i)\acute{\boldsymbol{h}}_k(i) + \acute{\boldsymbol{g}}_{\bar{k}}^T(i)\acute{\boldsymbol{g}}_{\bar{k}}(i)}}\right) \tag{24}$$

where $\acute{g}_k(i)$ is the element of $\acute{\boldsymbol{g}}_k(i)$ associated with the desired symbol $d_k(i)$, and $\acute{\boldsymbol{g}}_{\bar{k}}(i)$ is defined as $\acute{\boldsymbol{g}}_k(i)$ without the element associated with the desired symbol $d_k(i)$.

Averaging $P_{b,k}(i)$ over a large number (say 10 000) of symbol intervals will then give a good approximation of the BER averaged over time. For time-invariant systems, the technique can be simplified since $\boldsymbol{G}_i$ is constant at steady state, and time averaging of $P_{b,k}(i)$ will not be required.

V. Simulation Results

To verify the BER expressions (23) and (24), we simulated the Kalman smoother algorithm with four users, a processing gain of 8, and a detection delay arbitrarily selected to be $P = 3.875$, and measured the BER. Randomly selected short codes and time delays were used. The results are plotted in Fig. 3, where the solid curve represents the BER calculated using both (23) and (24) because they are indistinguishable. The simulated curve falls almost exactly over the analytical curve, as expected.

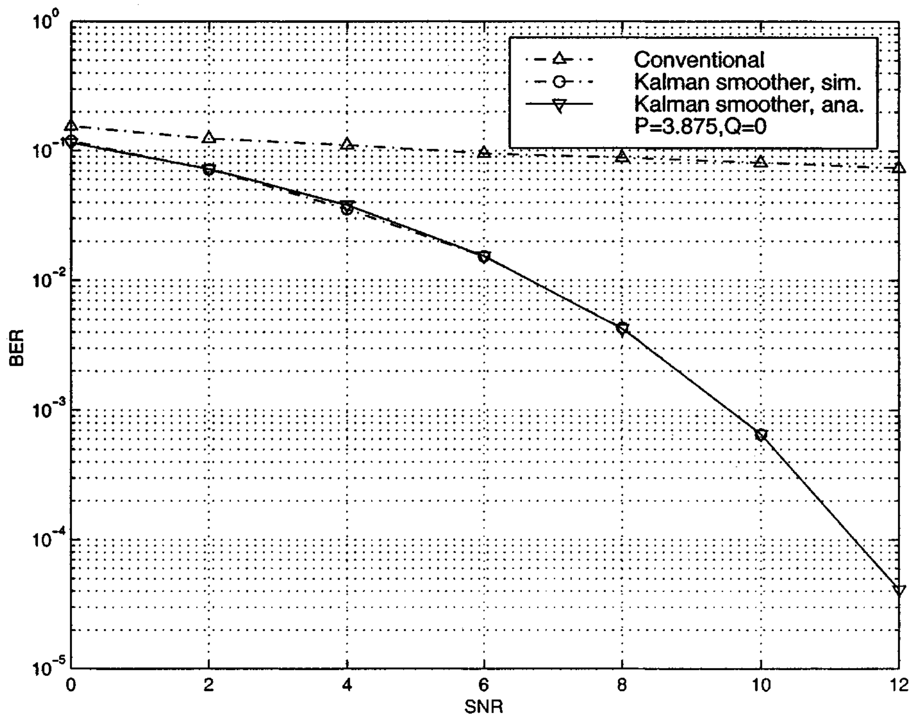

Fig. 3. Simulated and analytical BER curves for the Kalman smoother with $P = 3.875$.

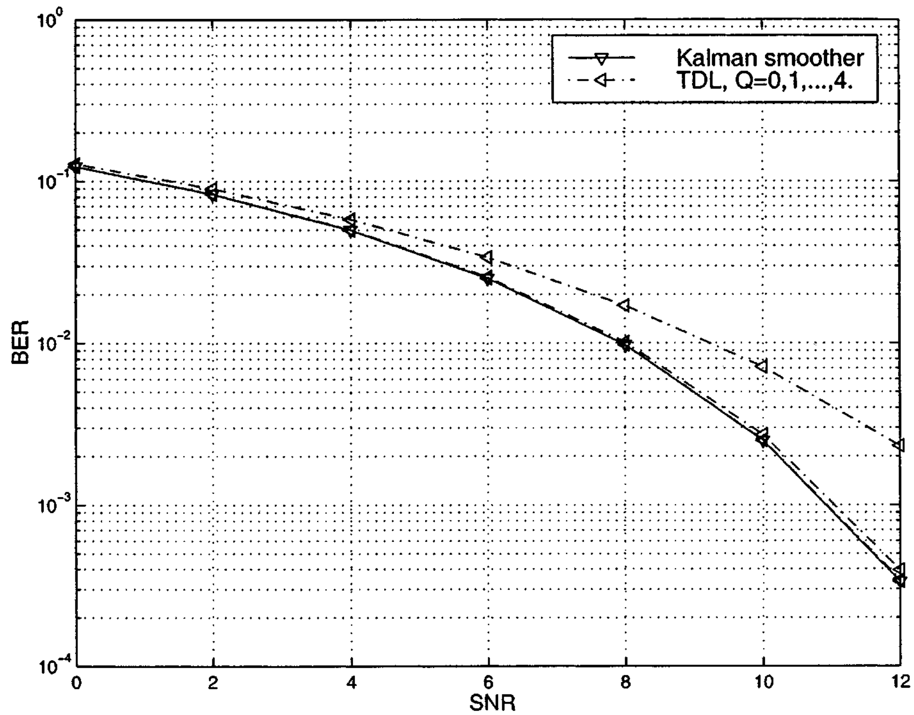

Fig. 4. Relative performance of the TDL and Kalman filter MMSE detectors for a detection delay $P = 0.875$, and varying observation window lengths.

The TDL MMSE detector is compared with the Kalman filter/smoother algorithm derived in this paper in Fig. 4. The system had five asynchronous users with randomly chosen time delays $\tau_k \in [0, N]$, oversampling rate $M = 1$, and processing gain $N = 8$. The detection delay was fixed at $P = 0.875$ symbols for both TDL and Kalman filter detectors, and the BERs were computed using the Gaussian assumption on the multiple-access interference. The observation window length in the TDL MMSE detector was varied by changing Q, defined in Section II. As expected, when the window span is small, the TDL detector does significantly worse than the Kalman filter detector. When the window size is increased, performance improves toward that of the Kalman filter detector but, of course, never exceeds it.

In Fig. 5, the effect of ignoring noise correlation when using overlapping observation windows is seen. The system parame-

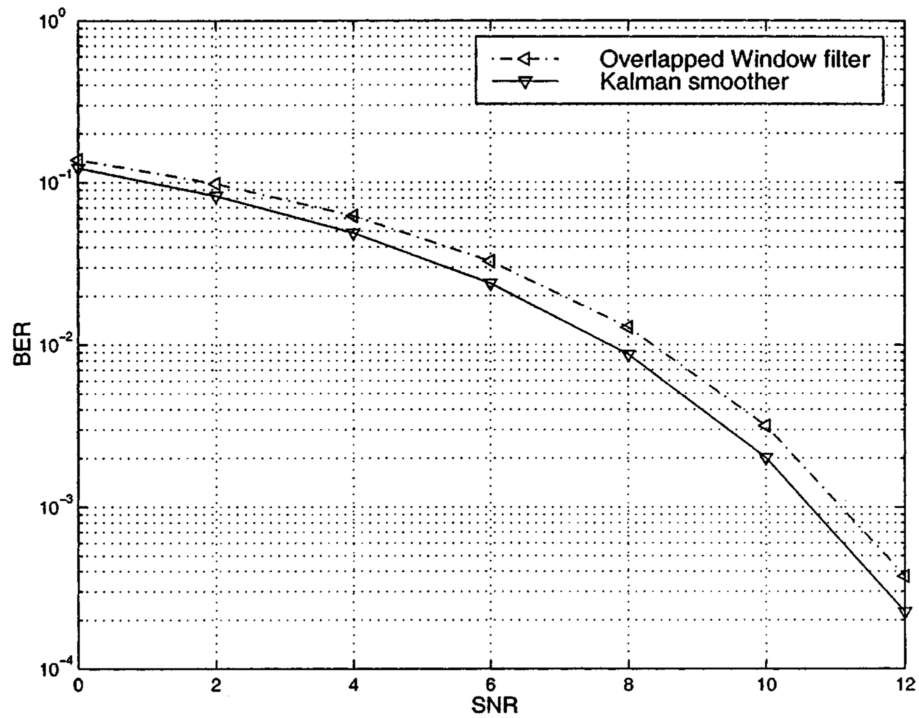

Fig. 5. Relative performance of the Kalman smoother (solid lines) and the Kalman filter ignoring noise correlations (dashed lines), with overlapping observation windows. P is fixed at 2.875.

ters are identical to those of the previous simulation, except that P is varied. The dashed lines represent the BER obtained using (6)–(9), i.e., assuming $\boldsymbol{n}(i)$ is a white-noise sequence although it is not, while the solid lines give the BER using the Kalman smoother of Section III-B, which no longer assumes white noise in $\boldsymbol{r}(i)$. Clearly, performance is improved when we use the correct model for the noise, thereby demonstrating the importance of the results of Section III-B.

VI. Conclusion

In this paper, we have shown that the Kalman filter or smoother is the exact or IIR linear MMSE detector for a given detection delay. Although it effectively accounts for all past information, the Kalman filter's complexity is fixed for a given detection delay, unlike in the case of the windowed LMMSE detector, which for the same performance will require a window whose length increases with time.

It was demonstrated that in time-invariant systems, the Kalman filter is a time-invariant filter, which allows one to use adaptive IIR filtering concepts in an implementation with training sequences in place of channel side information [10]. In addition, the BER of the Kalman detector in an AWGN channel can be computed semianalytically either with or without the Gaussian assumption on the MAI. The optimal MMSE BER can thus be obtained easily and used as a benchmark against which to compare new linear detector structures.

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