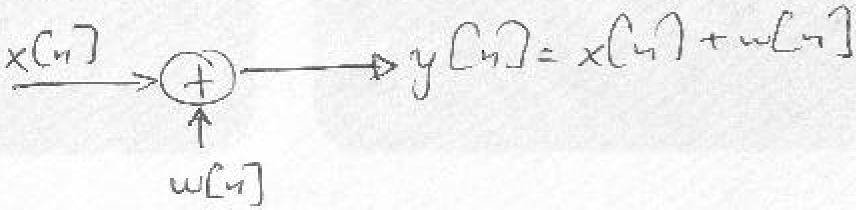
ECE 431 Tutorial 5

Random Signals & Noise

In many signal processing problems signals are corrupted by noise. In other words we consider the following additive noise model



where, y[n] is the observed signal, x[n] is the "desired" signal and w[n] is the corrupting noise. Usually, noise is "unpredictable" so we assume it is "random" and thus it can be treated "statistically". In most cases w[n] is assumed to be

- **white**: which means that samples of the noise are completely independent from each other. The frequency response of a white noise signal has a magnitude equal over all frequencies
- Gaussian: distributed by means of the Central Limit Theorem

or

- Uniformly distributed based on practical observations

$$w[n] \sim N(\mu_e, \sigma_w^2) \qquad w[n] \sim U[A, B]$$

Mean, variance — lower & upper limits

To treat statistically a random signal, we further make the following two assumptions.

1. The random signal or sequence is of infinite length
2. The random signal or noise is "stationary"; that is its statistics do not change with time.

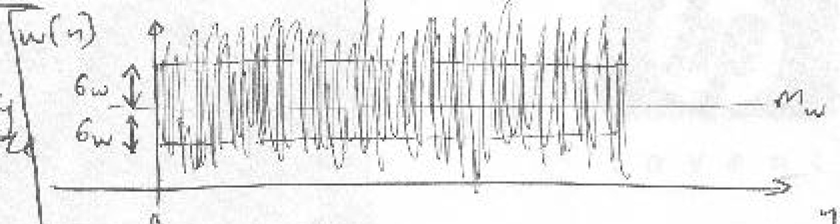
These assumptions allow us to calculate the "mean" and variance of the random sequence as follows:

mean: $$m_w = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} w[n]$$

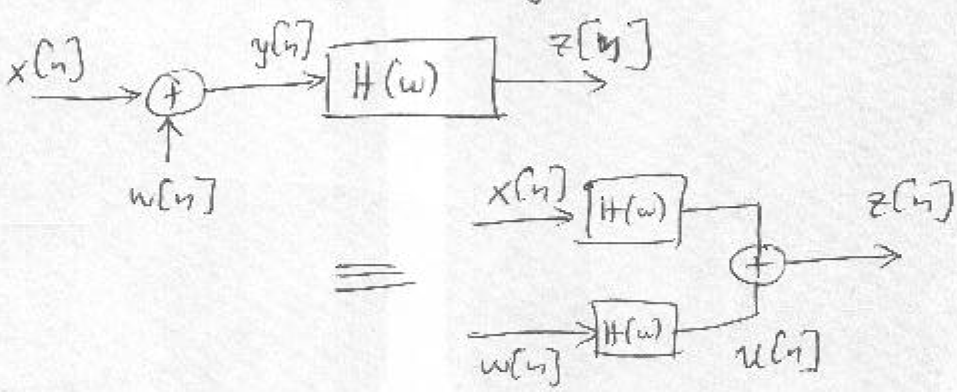
variance: $$\sigma_w^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left[w[n] - m_w \right]^2$$

Note:

For zero mean signals σ_w^2 is nothing else but the power of the signal



One important problem is to determine how the mean and variance of a random sequence $w[n]$ change with any linear transformation (e.g. filtering)



Let $h[n]$ be the impulse response of $H(\omega)$. That is $h[n] \xleftrightarrow{DTFT} H(\omega)$. Then, at the output of the filter

$$\rightarrow \boxed{u[n] = h[n] * w[n]} = \sum_{k=-\infty}^{\infty} h[k]\,w[n-k]$$

$$\rightarrow m_u = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \underbrace{\left(\sum_{k=-\infty}^{\infty} h[k]\,w[n-k]\right)}_{u[n]} = \sum_{k=-\infty}^{\infty} h[k] \underbrace{\left(\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} w[n-k]\right)}_{m_w}$$

Thus, $\boxed{m_u = m_w \left(\sum_{k=-\infty}^{\infty} h[k]\right)} \Rightarrow \boxed{m_u = m_w \cdot H(0)}$

$$\rightarrow \sigma_u^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left[\left(\sum_{k=-\infty}^{\infty} h[k]\,w[n-k]\right) - m_u\right]^2 = \cdots = \cdots$$

$$\boxed{\sigma_u^2 = \sigma_w^2 \left(\sum_{k=-\infty}^{\infty} h^2[k]\right)} \Rightarrow \sigma_u^2 = \sigma_w^2 \cdot \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\omega)|^2\, d\omega\right)$$

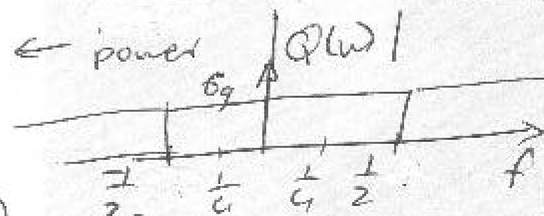
Parseval's Relation

Example: (Power of quantization noise)

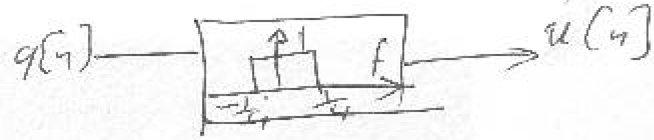
Given B bits to represent a discrete value, we know that the quantization noise $\rightarrow q[n] \sim U\left[-\frac{V_s}{2^B}, \frac{V_s}{2^B}\right]$

Thus, $m_q = 0$, $\sigma_q^2 = \frac{V_s^2}{3 \cdot 2^{2B}}$ ← power

Since, the noise is white



Let us low pass filter $q[n]$ with an ideal LP filter at $f_c = \frac{1}{4}$. Then, the power of the quantization noise at the output of the filter is

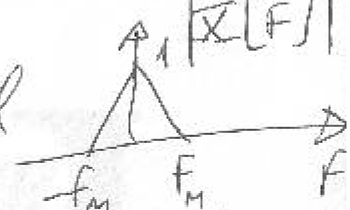


$$\sigma_u^2 = \sigma_q^2 \int_{-\frac{1}{4}}^{\frac{1}{4}} 1\, df = \sigma_q^2 \cdot \frac{2}{4} = \frac{\sigma_q^2}{2}$$

Also, $\mu_u = \mu_q \cdot H(0) = 0 \cdot H(0) = 0$

Thus, the power of the filtered noise has been reduced by 2.

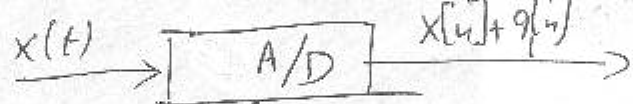
<u>Example:</u> Sampling frequency and Number of bits



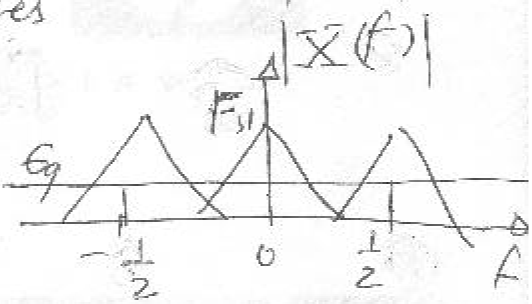
Let $x[t]$ be a bandlimited signal

Consider the following two cases

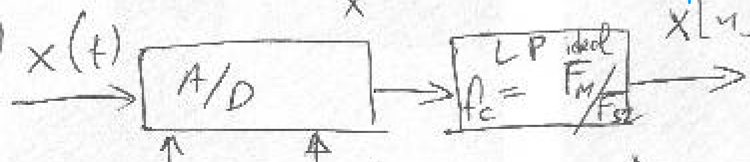
(A)



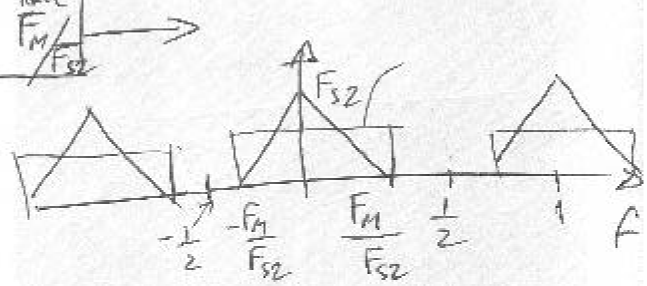
$F_{S1} = 2F_M$, B_1 bits



(B)



$F_{S2} \gg 2F_M$, B_2 bits



To have the same quantization noise power in the two cases

$$\frac{V_s^2}{3 \cdot 2^{2B_1}} = \frac{V_s^2}{3 \cdot 2^{2B_2}} \cdot \frac{2F_M}{F_S} \Rightarrow B_2 = B_1 + \frac{1}{2}\log_2 \frac{2F_M}{F_{S2}} = B_1 + \frac{1}{2}\log_2 \frac{F_{S1}}{F_{S2}} < B_1$$

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