Realization of FIR Filter using Frequency-Sampling Structure

Recall: In a previous lecture, you saw how the frequency-sampling method could be used to DESIGN FIR filters, i.e., given filter specifications, how to design a filter that best approximates these specs.

Goal: In this tutorial, we look at a related problem: realization of FIR filter, i.e., given an impulse response, how do we implement it? It turns out that the frequency-sampling concept can also be applied in this context.

Method:

* Given an FIR filter with impulse response

$$h[n] = \begin{cases} h[0], h[1], \ldots, h[N-1] \\ 0, \quad \text{for} \quad n < 0 \text{ and } n \geq N \end{cases} \tag{1}$$

$$\Big\updownarrow \text{N-DFT}$$

$$H[k], \quad k = 0, 1, \ldots, N-1 \tag{2}$$

* Consider the transfer function:

$$H(z) = \sum_{n=0}^{N-1} h[n] z^{-n} = \sum_{n=0}^{N-1} \Bigg[\underbrace{\frac{1}{N} \sum_{k=0}^{N-1} H[k] e^{\frac{jkn2\pi}{N}}}_{\text{IDFT}} \Bigg] z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} H(k) \left[\overbrace{\sum_{n=0}^{N-1} \left(e^{j\frac{2\pi k}{N}} z^{-1} \right)^{n}}^{\text{geometric series}} \right]$$

$$= \underbrace{\left[\frac{1}{N} \left(1 - z^{-N}\right) \right]}_{\text{(A)}} \underbrace{\left[\sum_{k=0}^{N-1} \frac{H(k)}{1 - e^{j2\pi k/N} z^{-1}} \right]}_{\text{(B)}} \qquad (3)$$

* Eq (3) suggests a cascade implementation of 2 parts :

(A) a comb filter

(B) parallel network or bank of resonators.

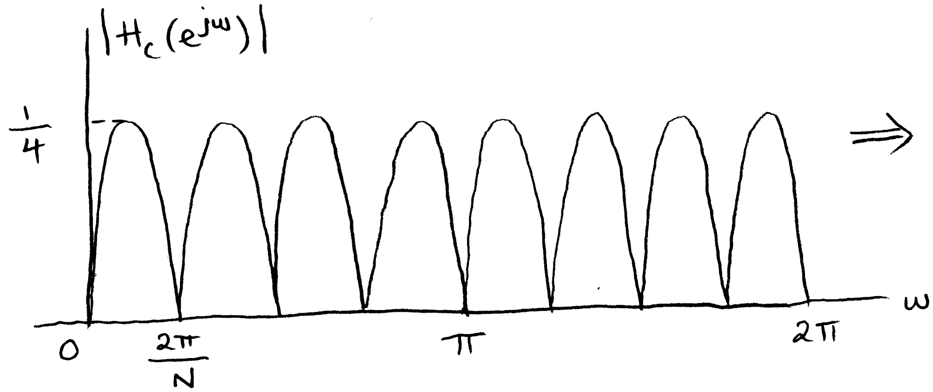
* <u>Comb filter</u> :

$$\frac{1 - z^{-N}}{N} = H_c(z) \text{ is basically an FIR filter} \qquad (4)$$

Also the freq. response :

$$H_c(e^{jw}) = \frac{1 - e^{-jNw}}{N} = \frac{2j}{N} e^{-jNw/2} \sin(Nw/2) \qquad (5)$$

For $N = 8$, the mag. response $|H_c(e^{jw})|$:



$\Rightarrow$ "comb"

* Resonators :

$$H_{R,k}(z) = \frac{H(k)}{1 - e^{j2\pi k/N} z^{-1}} \qquad (6)$$

is known as a resonator or a one-pole filter, with pole on the unit circle, at frequency $\omega_k = 2\pi k/N$.

$\Rightarrow$ (B) in (3) is a bank of N resonators

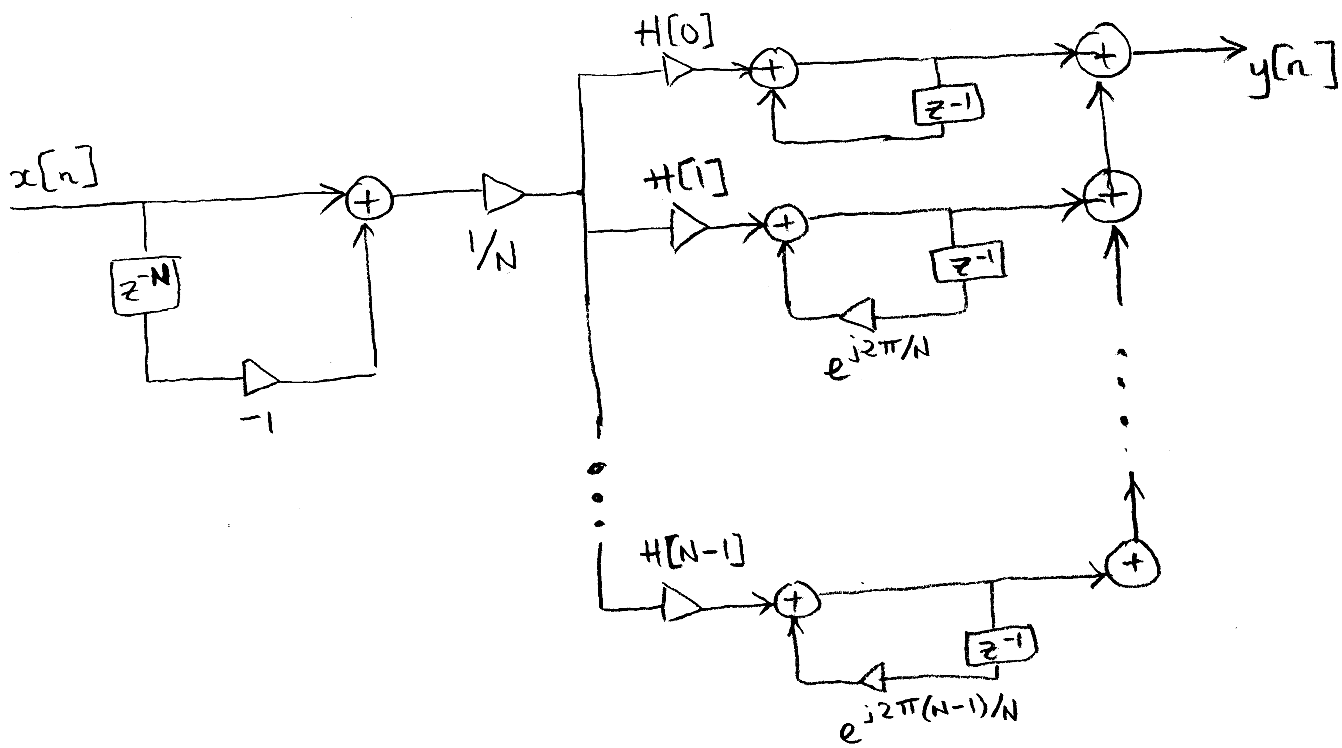
$\Rightarrow$ recursive implementation.

* Pole-zero cancellation :

The poles at $e^{j2\pi k/N}$ due to (B) in (3) are cancelled by the zeros due to (A), the comb filter, which are the roots of

$$1 - z^{-N} = 0 \qquad (7)$$

* Implementation : From (3), we have a cascade of :



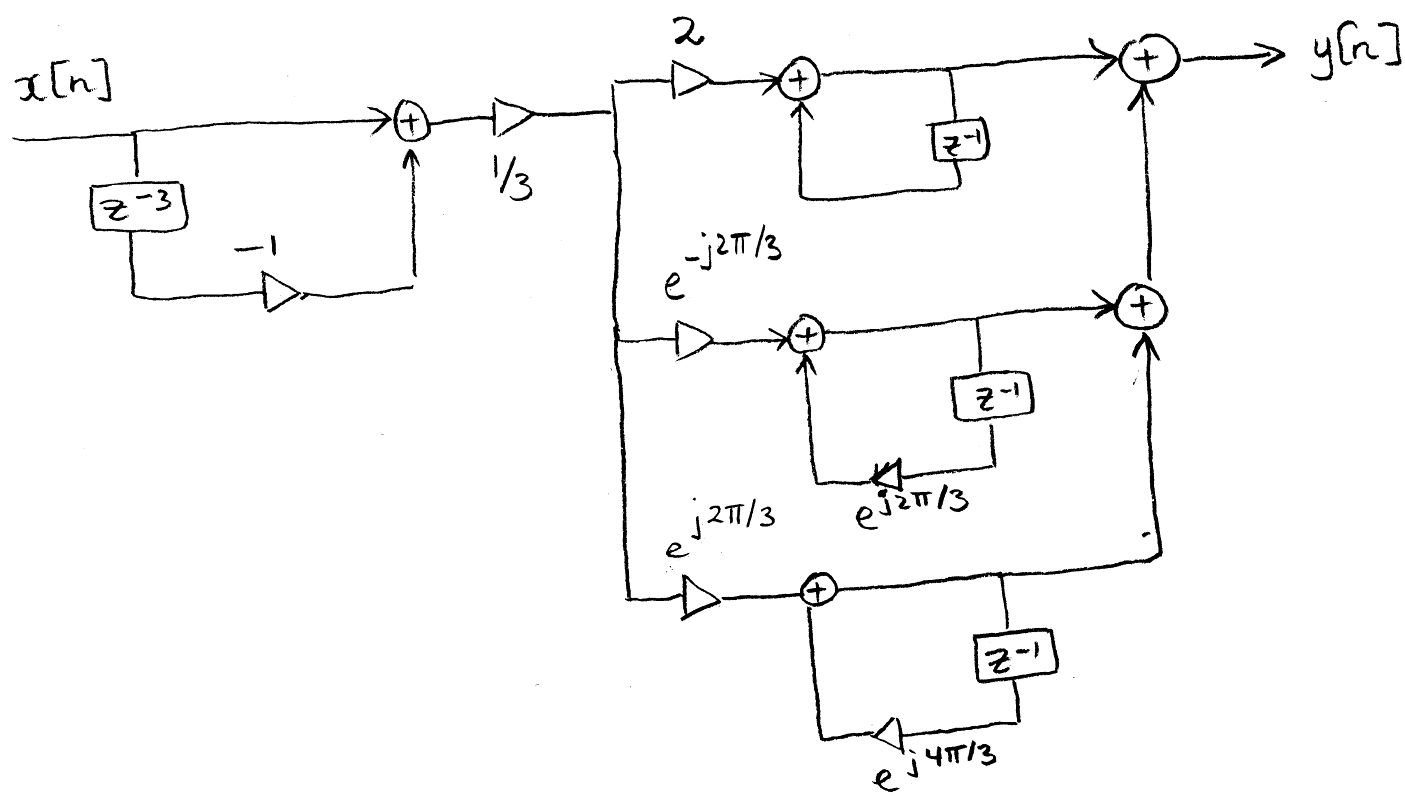
Example: Given an FIR filter with impulse response

$$h[n] = \begin{cases} 1/2 & , \; n = 0, 2 \\ 1 & , \; n = 1 \\ 0 & , \; \text{otherwise} \end{cases}$$

* Here $N = 3$; compute $3-\text{DFT}$:

$$H[k] = \left\{ 2, \; \frac{e^{-j2\pi/3}}{2}, \; \frac{e^{j2\pi/3}}{2} \right\}$$

$$\Rightarrow \; H(z) = \frac{1}{3}\left(1 - z^{-3}\right)\left[\frac{2}{1-z^{-1}} + \frac{\frac{e^{-j2\pi/3}}{2}}{1-e^{j2\pi/3}z^{-1}} + \frac{\frac{e^{j2\pi/3}}{2}}{1-e^{j4\pi/3}z^{-1}} \right] \qquad (8)$$



__Problem__ : the above example requires the use of __COMPLEX ARITHMETIC__

$\Longrightarrow$ Is real-valued processing possible?

Real-valued Realization

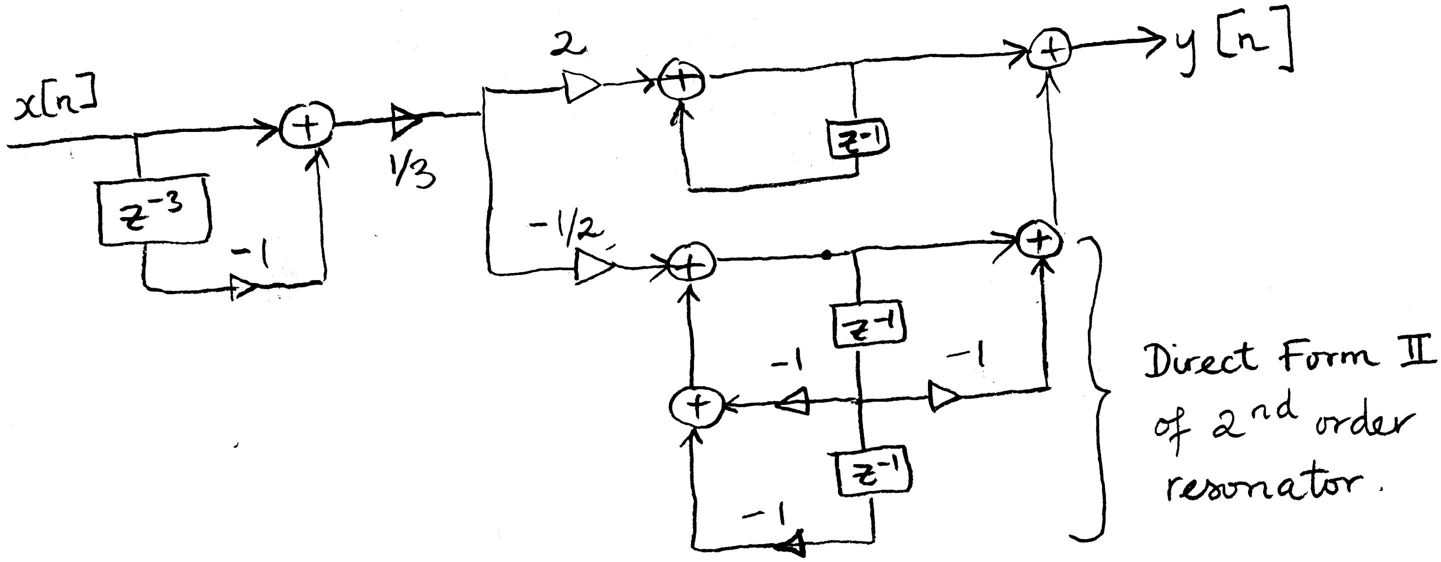
When $h[n]$ are real-valued for all n, it is possible and desirable to obtain a realization with real arithmetic
$\Rightarrow$ exploit the symmetry of the DFT and the $e^{j2\pi k/N}$ factor.

Example: continue with the previous example, we observe $H[2] = H[1]^*$, so that we can rewrite (8) as follows,

$$H(z) = \frac{1}{3}(1-z^{-3})\left[\frac{2}{1-z^{-1}} + \frac{\frac{e^{-j2\pi/3}}{2}}{1-e^{j2\pi/3}z^{-1}} + \frac{\frac{e^{j2\pi/3}}{2}}{1-e^{-j2\pi/3}z^{-1}}\right]$$

$$= \frac{1}{3}(1-z^{-3})\left[\frac{2}{1-z^{-1}} + \frac{\cos(2\pi/3) - \cos(4\pi/3)z^{-1}}{1-2\cos(2\pi/3)z^{-1}+z^{-2}}\right]$$

$$= \frac{1}{3}(1-z^{-3})\left[\frac{2}{1-z^{-1}} + \underbrace{\frac{-\frac{1}{2}(1-z^{-1})}{1+z^{-1}+z^{-2}}}_{\text{IIR} \rightarrow 2^{nd}\text{ order resonator}}\right]$$



Filter Design with Comb and Resonator Cascade:

Comb and resonator filters have simple structures and are implemented with relative ease. Interestingly, they also facilitate filter design. While a frequency-sampling structure can approximate any given filter (as we saw in the previous sections), it is particularly simple when designing specific lowpass, highpass or bandpass filters. Basically, given a comb filter, we select an appropriate cascade of resonators to construct the type of filter we desire. In the sequel, we shall see how to construct a simple bandpass filter using only comb and resonator filters.

Example A: (*Bandpass filter*) Given a comb filter with N = 16, so that

$$H_C(z) = \frac{1 - z^{-16}}{N} \Rightarrow H_C(e^{jw}) = \frac{1 - e^{-jw16}}{16}$$

Hence, there are 16 lobes in the magnitude response, and 16 zeros located at k2π/N, k=0,..,15 (see Fig. 1(a) below). The idea is to perform zero cancellation with a pole, using a resonator. Let us use a resonator with k=4, and a gain G_0=200,

$$H_{R,4}(z) = \frac{G_0}{1 - e^{j2\pi 4/16} z^{-1}}$$

Hence, the pole or resonating frequency is at k2π/N (see Fig. 1(b)). Now, cascading it with the previous comb

$$H_1(z) = H_C(z) H_{R,4}(z)$$

results in Fig. 1(c). Observe that the peak is precisely where the pole-zero cancellation occurs, and that basically this has "fused" two adjacent lobes (in the original comb filter) to create a new passband. Hence, using a -3dB bandwidth criterion, the passband of the cascade is approximately the bandwidth of the original lobe = 2π/N. Also, verify that the peak gain of the cascade is in fact G_0.

Next, recalling the property of the FT, we still need a conjugate passband, which is easily achieved by a conjugate resonator

$$H_1(z) = H_C(z)\big(H_{R,4}(z) + H_{R,N-4}(z)\big) = \frac{1 - z^{-16}}{16}\left(\frac{200}{1 - e^{j2\pi 4/16} z^{-1}} + \frac{200}{1 - e^{j2\pi 12/16} z^{-1}}\right)$$

Refer to Fig. 1(d,e) for the resulting magnitude responses.

In the above design, the parameter *N* controls not only the bandwidth, but also how accurately we can select the passband resolution. The larger *N* is the more zeros and lobes there are, so that there are more candidates for passbands. In fact, we can even select multiple passbands quite easily, as demonstrated in the next example.

Example B: (*Filter with 2 passbands*) We construct a more complex filter, with 2 different passbands each with a different peak gain, by using only a single comb filter and 2 pairs of resonators. Note that peak gains for the passbands can be selected independently, because they occur at the zeros of the comb filter.

Suppose we use a comb filter with N=32, and resonators at k=7, 32-7, 15, 32-15, with gains 100 and 400. Then Fig. 2 shows the resulting magnitude responses. As an exercise, see if you can write the expressions for the frequency responses for this example.

Note: Also, think about how you can realize lowpass and high-pass with the above procedure (we actually have solved the more challenging bandpass case). You can experiment with these designs quite easily in MATLAB, by trying different combinations of the parameters N, k.

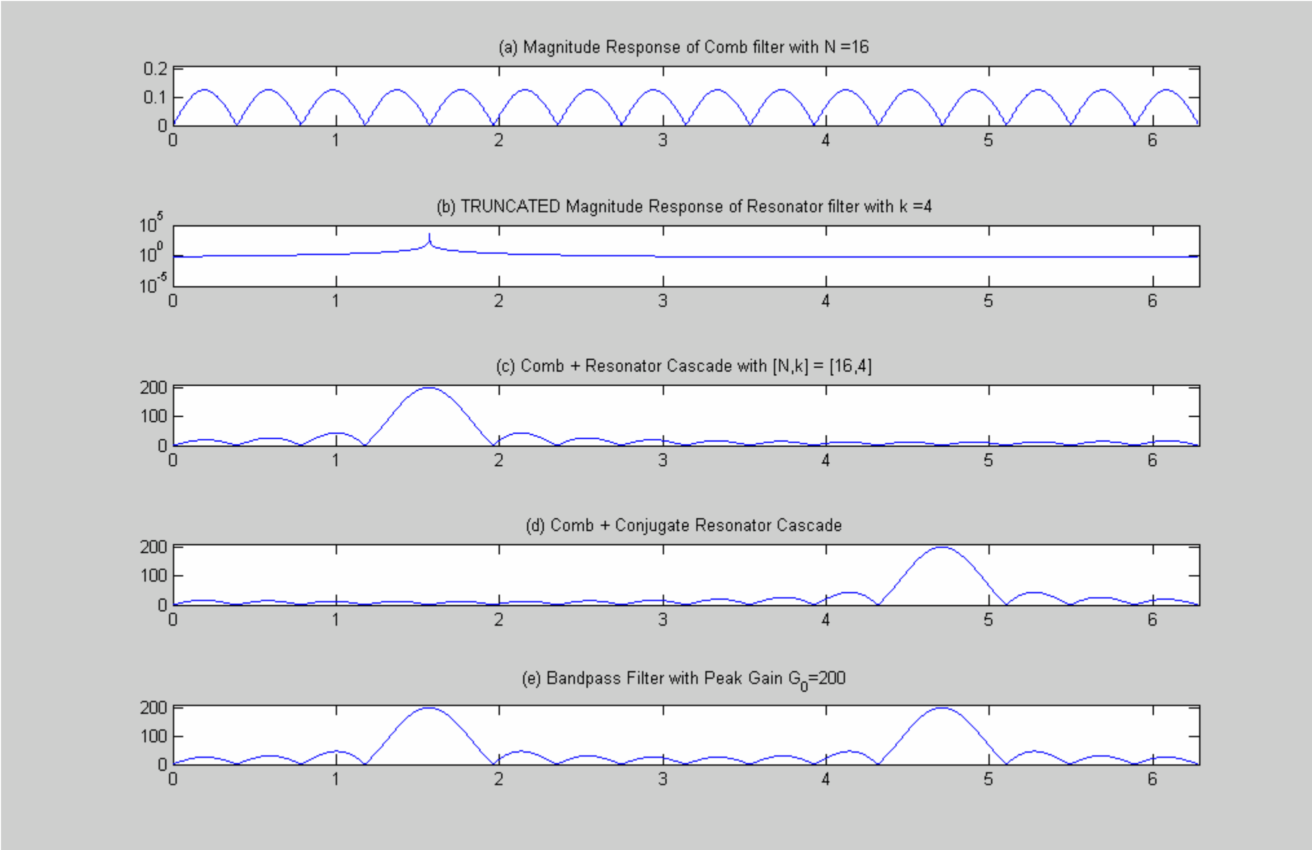


Figure 1 - Example A. Bandpass Filter

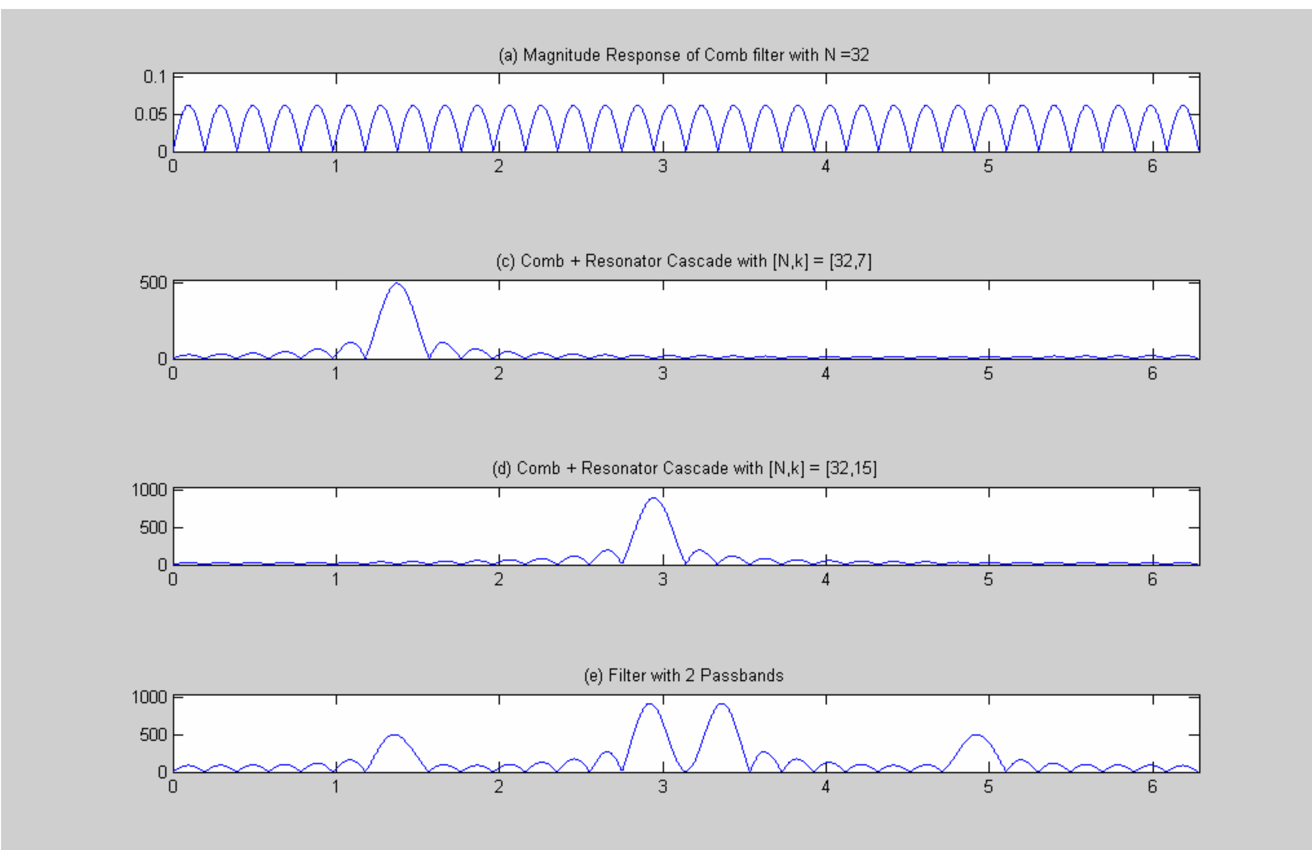


Figure 2 - Example B. Filter with 2 passbands