Complex Analytic Signals of Real Sequences

Recall : In the continuous-time domain, an analytic signal has spectrum equal 0 for all negative frequencies, i.e., $\mathcal{F}\{y(t)\} = Y(j\Omega) = 0$ for $-\infty < \Omega < 0$.
(Applications include single-sideband communication and analysis of bandpass systems in general).

Extension to discrete-time :

Same concept applies, but the meaning of positive vs. negative frequencies must be reconsidered.

$$\Rightarrow \quad Y(e^{j\omega}) = 0 \quad \text{for} \quad -\pi \le \omega < 0. \qquad (1)$$

Definition : For a real sequence $x[n]$, a complex analytic signal is constructed as:

$$y[n] = x[n] + j\,\hat{x}[n] \qquad (2)$$

where $\hat{x}[n]$ is also a real sequence.

* We wish to construct $\hat{x}[n]$ such that (1) is satisfied. It turns out that an extremely useful construction for such an $\hat{x}[n]$ is the (discrete) Hilbert transform.

* First, consider the following DTFT analysis of (2):

$$Y(e^{j\omega}) = X(e^{j\omega}) + j\,\hat{X}(e^{j\omega}) \qquad (3)$$

where :

$$\begin{aligned} x[n] &\longleftrightarrow X(e^{j\omega}) \\ \hat{x}[n] &\longleftrightarrow \hat{X}(e^{j\omega}) \end{aligned} \qquad (4)$$

* Now since $x[n]$ and $\hat{x}[n]$ are real, the DTFTs are conjugate symmetric:

$$X(e^{+j\omega}) = X^*(e^{-j\omega}) \quad \text{and} \quad \hat{X}(e^{+j\omega}) = \hat{X}^*(e^{-j\omega}) \tag{5}$$

* Thus, from (3) & (5)

$$X(e^{j\omega}) = \frac{1}{2}\left[Y(e^{j\omega}) + Y^*(e^{-j\omega})\right] \tag{6}$$

$$\text{and} \quad j\hat{X}(e^{j\omega}) = \frac{1}{2}\left[Y(e^{j\omega}) - Y^*(e^{-j\omega})\right] \tag{7}$$

* Furthermore, to satisfy (1), we arrive at:

$$\boxed{Y(e^{j\omega}) = \begin{cases} 2X(e^{j\omega}), & 0 \le \omega < \pi \\ 0, & -\pi \le \omega < 0 \end{cases}} \tag{8}$$

Hence, an analytic signal has a spectrum which is zero for negative frequencies, and twice the original spectrum for positive frequencies.

Characterization of $\hat{x}[n]$:

Observe that

$$Y(e^{-j\omega}) = 0 \quad \text{for } 0 \le \omega < \pi$$
$$\text{and} \quad Y(e^{j\omega}) = 0 \quad \text{for } -\pi \le \omega < 0 \tag{9}$$

Then (7) can be rewritten as:

$$\hat{X}(e^{j\omega}) = \begin{cases} -jX(e^{j\omega}), & 0 \le \omega < \pi \\ jX(e^{j\omega}), & -\pi \le \omega < 0 \end{cases} \tag{10}$$
$$= X(e^{j\omega})\, \hat{H}(e^{j\omega})$$

where :

$$\hat{H}(e^{jw}) = \begin{cases} -j \ , & 0 \leq w < \pi \\ j \ , & -\pi \leq w < 0 \end{cases} \qquad (11)$$

is known as the Hilbert transformer, and the output $\hat{x}[n]$ the Hilbert transform of the input signal $x[n]$.

* The impulse response $\hat{h}[n]$ of the Hilbert transformer can be obtained using the inverse DTFT :

$$\hat{h}[n] = \begin{cases} 0 \ , & n \text{ even} \\ \frac{2}{\pi n} \ , & n \text{ odd} \end{cases}$$

which is a two-sided infinite-length impulse response, and is thus unrealizable. Some approximation is then needed in actual implementation.

Practical Generation of Analytic Signals using DFT

Re-examining (8), we see that $y[n]$ can be generated by passing $x[n]$ through :

$$H(e^{jw}) = \begin{cases} 2 \ , & 0 \leq w < \pi \\ 0 \ , & -\pi \leq w < 0 \end{cases} \qquad (12)$$

So that $Y(e^{jw}) = H(e^{jw})\, X(e^{jw})$ (13)

produces (8).

* Now, for finite-length sequences, the discrete FT can be used (which has an efficient implementation FFT). This corresponds basically to sampling in the frequency domain.

* Algorithmically, to obtain $y[n]$ from a sequence $x[n]$, $n = 1, 2, \cdots, N$

① Compute $X[k] = FFT\{x[n]\}$, $k = 1, 2, \cdots, N$

② Form:

$$Y[k] = \begin{cases} 2X[k] & , \ k = 2, 3, \cdots, N/2 \\ X[k] & , \ k = 1, \frac{N}{2}+1 \\ 0 & , \ k = \frac{N}{2}+2, \cdots, N \end{cases}$$

③ Obtain $y[n] = iFFT\{Y[k]\}$, $n = 1, 2, \cdots, N$

<u>Example</u>: $x[n] = \{-2, 0, 3, 0\}$

$\xrightarrow{FFT}$ $X[k] = \{1, -5, 1, -5\}$

$\longrightarrow$ $Y[k] = \{1, -10, 1, 0\}$

$\xrightarrow{iFFT}$ $y[n] = \{-2, -2.5j, 3, 2.5j\}$

Then $y[n] = x[n] + j\,\hat{x}[n]$ (14)

is an analytic signal.

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Appendix

(*) The observant reader is likely to note that the given DFT-based generation algorithm does NOT correspond exactly to a simple sampling of (12).

* The value of $Y[1] = X[1]$ instead of $2X[1]$ ($H(e^{j\omega}) = 2$ for $\omega = 0$ according to (12)).
  This deliberate modification is more suited to the behaviour of the FT at a discontinuity : it converges to the average of the values on either side of the discontinuity, i.e., $\frac{2X[1] + 0}{2} = X[1]$

* The value of $Y[\frac{N}{2}+1] = X[\frac{N}{2}+1]$ rather than $2X[\frac{N}{2}+1]$.
  The omission of the factor 2 is so that the resulting signal $y[n] = y_R[n] + j\, y_I[n]$ has characteristics of an analytic-like discrete-time signal :
  1) real part = original discrete-time sequence
  $$y_R[n] = \mathrm{Re}\{y[n]\} = x[n], \quad 1 \le n \le N$$
  2) real and imaginary components must be orthogonal over the finite interval.
  $$\sum_{n=1}^{N} y_R[n]\, y_i[n] = 0$$

* For other characteristics and pitfalls, see the paper :
  S. L. Marple " Computing the discrete-time Analytic Signal via FFT ", IEEE Trans. Signal Processing, Vol 47, No. 9, Sep 1999.