Digital frequency oscillators

Trigonometric Identity

$$\boxed{\sin(w_0 n) = 2\cos(w_0)\cdot\sin(w_0(n-1)) - \sin(w_0(n-2))}$$

$$|w_0| \le \pi$$

Proof: $\sin(w_0 n) = \sin(w_0(n-1) + w_0) =$

$$= \sin(w_0(n-1))\cos(w_0) - \sin(w_0)\cos(w_0(n-1))$$

$$= 2\cos(w_0)\sin(w_0(n-1)) - [\cos(w_0)\sin(w_0(n-1))$$

$$+ \sin(w_0)\cos(w_0(n-1)) =$$

$$= 2\cos(w_0)\sin(w_0(n-1)) - \sin(w_0(n-1) - w_0)$$

So now let: $y[n] = \sin(w_0 n)$, then

$$\boxed{y[n] = 2\cos(w_0)\, y[n-1] - y[n-2]}$$ AR(2) model !!

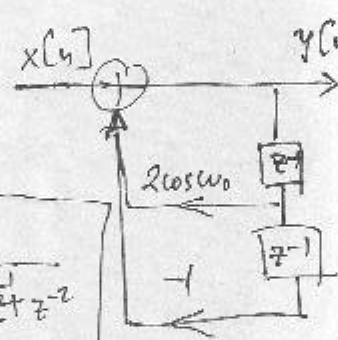
So we can model $y[n]$ as an AR(2) model ($a_1 = -2\cos w_0$, $a_2 = 1$) with initial conditions, i.e.

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = \underbrace{x[n]}_{i.c.}$$

This model has two poles at

$e^{\pm j w_0}$ (on the unit circle)

$$\boxed{\frac{Y(z)}{X(z)} = \frac{1}{1 - 2\cos w_0 z^{-1} + z^{-2}}}$$

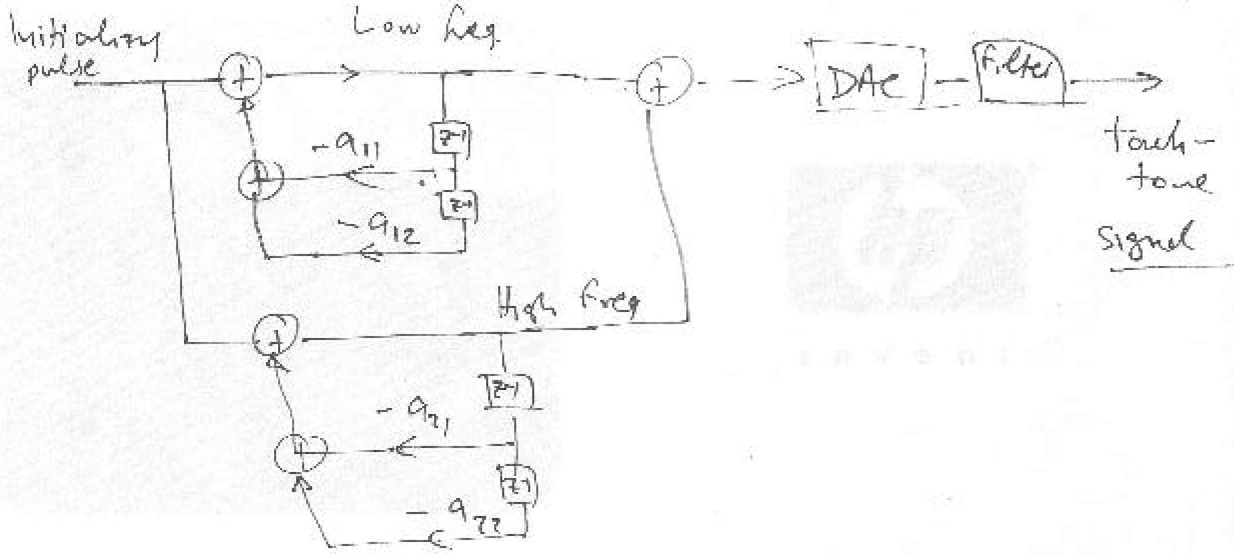


Digital Telephony: Dual tone multifrequency (DTMF)

	1209 Hz	1336 Hz	1477	1633 Hz
697 Hz	1	2	3	A
770 Hz	4	5	6	B
852 Hz	7	8	9	C
941 Hz	*	0	#	D

Sampling frequency 8 KHz

* Each digit is represented by a combination of low and high frequency



Example: Find the coefficients of the two oscillators for the digit "9"

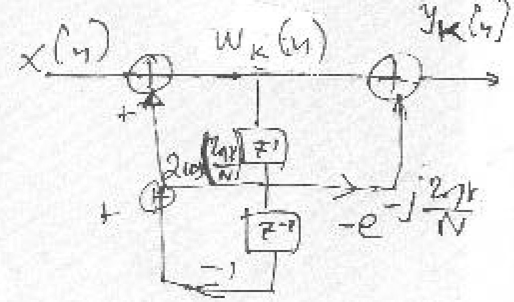
$\omega_1 T = 852$ Hz, so $a_{11} = 2\cos(\omega_1 T) = 2\cos\left(2\pi\frac{852}{8\cdot 10^3}\right) = 1.5687$

$\omega_2 T = 1477$ Hz

$a_{21} = 2\cos(\omega_2 T) = 2\cos\left(2\pi\frac{1477}{8\cdot 10^3}\right) = 0.7986$

<u>The Goertzel Algorithm</u>: This is a high Q, narrowband filter, second order IIR having transfer function

$$H_k(z) = \frac{1 - e^{-j\frac{2\pi k}{N}} z^{-1}}{1 - 2\cos\left(\frac{2\pi k}{N}\right) z^{-1} + z^{-2}},$$



$$= \frac{1}{1 - e^{j\frac{2\pi k}{N}} z^{-1}}$$

So the LCCDEs: $\left[\begin{array}{l} w_k(n) = 2\cos\left(\frac{2\pi k}{N}\right) w_k(n-1) - w_k(n-2) + x(n) \\ y_k[n] = w_k[n] - e^{-j\frac{2\pi k}{N}} w_k(n-1) \end{array}\right]$

with $w_k(-1) = w_k(-2) = 0$

Also,

$$|y_k[n]|^2 = y_k(n)\, y_k^*(n) = \left(w_k(n) - e^{-j\frac{2\pi k}{N}} w_k(n-1)\right) \cdot \left(w_k^*(n) - e^{j\frac{2\pi k}{N}} w_k^*(n-1)\right) = |w_k[n]|^2 - e^{-j\frac{2\pi k}{N}} w_k(n-1) w_k^*(n) - e^{j\frac{2\pi k}{N}} w_k^*(n-1) w_k(n) + |w_k(n-1)|^2 =$$

$$= |w_k(n)|^2 + |w_k(n-1)|^2 - 2\cos\left(\frac{2\pi k}{N}\right) w_k(n) w_k(n-1)$$

Usually, the value of $|y_k[n]|_{n=N}$ is used for detection of the incoming frequency. Thus for detecting the frequency at bin k only one real coefficient $2\cos\left(\frac{2\pi k}{N}\right)$ is needed.

Derivation of the Goertzel Algorithm

N-DFT: $$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}}$$

$$= \underbrace{e^{j\frac{2\pi k N}{N}}}_{1} \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{+j\frac{2\pi k (N-n)}{N}}$$

$$= x(n) * h_k(n) \Big|_{n=N} = y_k(N)$$

where the filter $h_k(n) = e^{j\frac{2\pi k n}{N}} u(n)$

Then $$H_k(z) = \frac{1}{1 - e^{j\frac{2\pi k}{N}} z^{-1}} = \frac{1 - e^{-j\frac{2\pi k}{N}}}{1 - 2\cos\left(\frac{2\pi k}{N}\right) z^{-1} + z^{-2}}$$

Thus, this filter at $n=N$ calculates $X(k) = y_k(N)$