ARRAY PROCESSING

• Introduction to Array Processing

Signal subspace and noise subspace  Array Processing methods

• Methods for resolving coherent sources

Wideband Array Processing methods
INTRODUCTION TO ARRAY PROCESSING (1)

- Array is a set of sensors. The output signals from the sensors are combined appropriately to produce a desired result.

\[ x_1(n), \ldots, x_7(n) \] is a "snapshot"
INTRODUCTION TO ARRAY PROCESSING (2)

Narrowband Array Processing  Wideband Array Processing

SOURCE SPECTRUM

■ APPLICATIONS

1) SONAR
2) RADAR
3) GEOPHYSICS
4) BIOMEDICINE
INTRODUCTION TO ARRAY PROCESSING (3)

- **ARRAY CONFIGURATIONS**

  - LINEAR
  - CIRCULAR
  - PLANAR
  - T-SHAPE

- "SNAPSHOT": is a set of sensor readings (outputs) at time instant $n$

- **ARRAY PROCESSING**: is the Power Spectral Analysis along snapshots (instead of time)

- **1-d ARRAY PROCESSING**: A snapshot is in the form of 1-d signal
INTRODUCTION TO ARRAY PROCESSING

■ 1-d ARRAY PROCESSING

- Difficult spectral estimation problem since the number of sensors is usually small. However, we can use "ensemble" averaging over different snapshots.
1-d ARRAY PROCESSING

\[ x_m(n) = \sum_{i=1}^{I} s_i [u(i,m) + \phi(i,n)] + w_m(n) \]

- \( x_m(n) \): \( m^{th} \) sensor at time \( n \)
- \( s_i \): phase difference
- \( u(i,m) \): random phase noise
- \( \phi(i,n) \): sensor at time \( n \)
- \( w_m(n) \): phase difference
- \( \theta_i \): bearing of source \( s_i(.) \)
- \( \lambda_i \): wavelength of source \( s_i(.) \)
- \( d \): spacing between sensors
INTRODUCTION TO ARRAY PROCESSING (6)

1-d ARRAY PROCESSING

CIRCULAR ARRAY

MODEL EQUATION (I incoherent sources)

\[ x_m(n) = \sum_{i=1}^{I} s_i [u(i,m) + \phi(i,n)] + w_m(n) \]

- \( m^{th} \) sensor at time \( n \)
- phase difference
- random phase noise

\[ u(i,m) = \frac{2\pi R}{\lambda_i} \left[ \cos\left(\frac{2\pi m}{M} - \theta_i\right) - \cos\left(\frac{2\pi m}{M}\right) \right] \]
1-d ARRAY PROCESSING

Objective: Given \( \{x_m(n)\} \), \( n=1,...,N \), \( m=1,...,M \), estimate \( \{\theta_i\} \), \( i=1,...,I \)

DUALITY BETWEEN PSE and AP

1) \( P(f) \), \( -\frac{1}{2} \leq f \leq \frac{1}{2} \) ↔ \( P(\theta) \), \( -90 \leq \theta \leq 90^\circ \)

2) Frequency, \( f \) ↔ Angle, \( \sin \theta \)

3) Sampling period, \( T \) ↔ sensor spacing, \( d \)

4) Aliasing \( \left| f_o > \frac{1}{2T} \right| \) ↔ Aliasing \( [\lambda < 2d] \)
INTRODUCTION TO ARRAY PROCESSING (6)

1-d ARRAY PROCESSING

MODEL EQUATION (I incoherent sources)

\[ x_m(n) = \sum_{i=1}^{I} s_i[u(i,m) + \phi(i,n)] + w_m(n) \]

- \(m^{th}\) sensor at time \(n\)
- phase difference
- random phase noise

\[ u(i,m) = \frac{2\pi R}{\lambda_i} \left[ \cos \left( \frac{2\pi m}{M} - \theta_i \right) - \cos \left( \frac{2\pi m}{M} \right) \right] \]
INTRODUCTION TO ARRAY PROCESSING (7)

- **1-d ARRAY PROCESSING**

  - **Objective:** Given \( \{ x_m(n) \} \), \( n=1,...,N \), \( m=1,...,M \), estimate \( \{ \theta_i \} \), \( i=1,...,I \)

- **DUALITY BETWEEN PSE and AP**

  1) \( P(f) \), \( -\frac{1}{2} \leq f \leq \frac{1}{2} \) ↔ \( P(\theta) \), \( -90 \leq \theta \leq 90^\circ \)

  2) Frequency, \( f \leftrightarrow \) Angle, \( \sin \theta \)

  3) Sampling period, \( T \leftrightarrow \) sensor spacing, \( d \)

  4) Aliasing \( \left[ f_o > \frac{1}{2T} \right] \) ↔ Aliasing \( [\lambda<2d] \)
1-d ARRAY PROCESSING

Estimate $P(\theta) \sim \theta$ or equivalently find the distribution of power for the set $\{x_m(n)\}$ on the exponentials:

$$\left\{ e^{\frac{j2\pi(m-1)d\sin\theta_i}{\lambda}} \right\}, \quad m = 1, 2, \ldots, M$$

Notice the similarity with Fourier transform exponentials:

$$e^{j2\pi fnT}, \quad n = 1, 2, \ldots$$

Actually, array processing is nothing else but evaluation of the Fourier transform along snapshots of the array.

AMBIGUITY: To avoid distributing the power on $e^{\frac{j2\pi(m-1)d\sin\theta_i}{\lambda}}$ (sign ambiguity), compute first the analytical signal for each sensor signal $\{x_m(n)\}, \quad n=1, 2, \ldots, N$
INTRODUCTION TO ARRAY PROCESSING (9)

COMPLEX ANALYTIC SIGNAL FOR SENSOR $m$

$$y_m(n) = x_m(n) + jH[x_m(n)]$$  
where $H[.]$ is the Hilbert transform

COMPUTATION USING FFT

Given $\{x_m(n)\}$, $n=1,2,\ldots,N$

1) Obtain: $X_m(k) = \text{FFT}[x_m(n)], \quad k=1,\ldots,N$

2) Form: $Y_m(k) = \begin{cases} 
X_m(k), & k = 2,3,\ldots, \frac{N}{2} \\
X_m(k)/2, & k = 1, \frac{N}{2} + 1 \\
0, & k = \frac{N}{2} + 2,\ldots,N
\end{cases}$

3) Obtain: $y_m(n) = \text{IFFT} [Y_m(k)], \quad n=1,\ldots,N$
INTRODUCTION TO ARRAY PROCESSING (10)

■ SPATIAL COVARIANCE MATRIX OF SNAPSHOTS

\[ R = E \{ Y(n) Y^H(n) \} \]

\( M \times M \quad M: \# \text{ of sensors} \)

\[ Y^T(n) = [y_1(n), y_2(n), \ldots, y_M(n)]^T \]

■ ESTIMATED COVARIANCE MATRIX

\[ \hat{R} = \frac{1}{N} \sum_{n=1}^{N} Y(n) Y^H(n) \]

\( M \times M \)

(If \( N < M \), then \( \hat{R} \) is singular)
ARRAY PROCESSING METHODS (1)

- **CONVENTIONAL (BEAMFORMING)**

\[ P(\theta) = c^H \hat{R} c \]

- **ML OF CAPON**

\[ P(\theta) = \frac{1}{c^H \hat{R}^{-1} c} \]

- **AR**

\[ P(\theta) = \frac{1}{|u^T \hat{R}^{-1} c|^2} \]

- **THERMAL NOISE**

\[ P(\theta) = \frac{1}{|u^T \hat{R}^{-H} \hat{R}^{-1} c|^2} \]
ARRAY PROCESSING METHODS (2)

- $u^T = [1, 0, 0, \ldots, 0] (1 \times M)$ (Unit steering vector)

- $c = [1, e^{j\phi(1)}, e^{j\phi(2)}, \ldots, e^{j\phi(M-1)}]^T$ (Steering vector)

\[ \phi(m) = \frac{2\pi d}{\lambda} (m-1) \sin \theta \quad \text{(LINEAR ARRAY)} \]

\[ \phi(m) = \frac{2\pi R}{\lambda} \left[ \cos \left( \frac{2\pi m}{M} - \theta \right) - \cos \left( \frac{2\pi m}{M} \right) \right] \quad \text{(CIRCULAR ARRAY)} \]

\[ \phi(m) = m\, 2\pi fT \quad \text{(SPECTRUM ESTIMATION)} \]
ARRAY PROCESSING METHODS (3)

RESOLUTION

CONV

\[ \Delta k \geq \frac{1}{dM} \]

ML

\[ \Delta k \geq \frac{1}{\sqrt{SNR}dM} \]

AR

\[ \Delta k \geq \frac{1}{SNR dM} \]
ARRAY PROCESSING METHODS (4)  

EXAMPLE: 3 SOURCES

ML

CONVENTIONAL

AR
ARRAY PROCESSING METHODS (5)

EXAMPLE: \[ M=8, \; N=1024, \; I=2, \; \text{BEARING } 18^\circ \text{ and } 22^\circ \]
ARRAY PROCESSING METHODS (6)

EFFECT OF THE CHOICE OF UNIT STEERING VECTOR

\[ u^T = [1, 0, 0, 0, 0, 0] \quad (m_0=1) \quad u^T = [0, 0, 0, 0, 0, 1] \quad (m_0=6) \]
\[ u^T = [0, 0, 0, 1, 0, 0] \quad (m_0=4) \]

LINEAR PREDICTION FORMULA

\[ x_{m_0}(n) = \sum_{m, m \neq m_0} a_m x_m(n), \]
\[ \text{minimize } E\{|x_{m_0}(n) - \hat{x}_{m_0}(n)|^2\} \]
ARRAY PROCESSING METHODS (7)

EFFECT OF DIFFERENT CHOICES FOR UNIT STEERING VECTOR

![Graph showing the effect of different choices for unit steering vector.

- (a) SNR = 0 dB
  - $m_0 = 1$
  - $m_0 = 4$

- (b) SNR = -10 dB
  - $m_0 = 1$
  - $m_0 = 4$]
ARRAY PROCESSING METHODS (8)

PROPERTIES OF SPATIAL COVARIANCE MATRIX

1) LINE ARRAY:

\[
\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & \cdots & M-1 & M
\end{array}
\]

2) I narrowband plane waves of frequency \( w_0 = 2\pi f_0 \) and bearings

\[ \theta_1, \theta_2, \ldots, \theta_I, \quad I < M; \quad d < \frac{\lambda_0}{2} \]

where \( \lambda_0 = \frac{c}{f_0} \);

c: the speed of propagation.

3) Sensor analytic signals (Linear array)

\[
y_m(n) = \sum_{i=1}^{I} s_i(n) e^{-j\omega_0(m-1)\frac{\sin\theta_i}{c}d} + w_m(n)
\]

\( s_i(.) \) is the signal of \( i^{th} \) wavefront

\( w_i(.) \) is additive noise with variance \( \sigma^2 \), uncorrelated with each other noise sources and with signals.
ARRAY PROCESSING METHODS (9) (noise sources assumed spatially uncorrelated)

4) In vector form \( Y(n) = \mathbf{A} \mathbf{S}(n) + \mathbf{W}(n) \)

\[
\begin{bmatrix}
M \times 1 & M \times 1 & I \times 1 & M \times 1
\end{bmatrix}
\]

\[
Y(n) = [y_1(n), y_2(n), \ldots, y_M(n)]^T \quad (M \times 1)
\]

\[
S(n) = [s_1(n), s_2(n), \ldots, s_I(n)]^T \quad (I \times 1)
\]

\[
A(n) = [\alpha_1(\theta_1), \alpha_2(\theta_2), \ldots, \alpha_M(\theta_I)]^T \quad (M \times I)
\]

where \( \alpha(\theta_i) = [1, e^{-j\phi_i(1)}, \ldots, e^{-j\phi_i(M-1)}]^T \)

\[
\phi_i(m) = \omega_0 \frac{d}{c} \cdot m \cdot \sin \theta_i = 2\pi \cdot m \cdot d \cdot \frac{\sin \theta_i}{\lambda_0}
\]

\[
W(n) = [w_1(n), w_2(n), \ldots, w_M(n)]^T
\]

With M sensors we can only identify up to M incoherent sources
ARRAY PROCESSING METHODS (10)

SPATIAL COVARIANCE MATRIX

\[ R \triangleq E \{ Y(n) Y^H(n) \} \]

Following a procedure similar to narrowband PSE method

\[ R = S_M + W_M \]

where,

\[ S_M = A S \]

\[ W_m = \sigma^2 I_m \quad \text{(White noise assumption)} \]
ARRAY PROCESSING METHODS (11)

SPATIAL COVARIANCE MATRIX

Thus, for \( R = S_M + W_M \);

\[ W_m = \sigma^2 I_M \]

We have:

\[ \text{Rank}(S_M) = I \quad \text{(NUMBER OF SOURCES)} \]

\[ \text{Rank}(W_M) = M \quad \text{(NUMBER OF ELEMENTS)} \]

\[ \text{Rank}(R) = M \]

If there is no noise \( (\sigma^2 = 0) \), then \( R = S_M \) and therefore \( \text{Rank}(R)=I \)
<table>
<thead>
<tr>
<th>${s_i(n)}, \ i = 1, \ldots, I$</th>
<th>$S = E{S(n) S^H(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$I \times I$</td>
</tr>
<tr>
<td>1) Uncorrelated</td>
<td>Diagonal non-singular</td>
</tr>
<tr>
<td>2) Partially correlated</td>
<td>Non-diagonal, non-singular</td>
</tr>
<tr>
<td>3) Coherent, fully correlated</td>
<td>Non-diagonal, singular</td>
</tr>
</tbody>
</table>

$$\left( \frac{\sin \theta_i}{\lambda_i} = \frac{\sin \theta_j}{\lambda_j} \right)$$
ARRAY PROCESSING METHODS (13)

- ORTHOGONAL DECOMPOSITION OF $R$

$$R = \sum_{i=1}^{M} \rho_i V_i V_i^H; \quad V_i^H V_i = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

- SIGNAL + NOISE EIGENVECTORS

$$\{ V_1, V_2, \ldots, V_I \} \quad \rho_i = \lambda_i + \sigma^2; \quad i = 1, \ldots, I$$

- NOISE EIGENVECTORS

$$\{ V_{I+1}, V_{I+2}, \ldots, V_M \} \quad \rho_i = \sigma^2; \quad i = I+1, \ldots, M$$

In other words:

$$R = \sum_{i=1}^{I} (\lambda_i + \sigma^2) V_i V_i^H + \sigma^2 \sum_{i=1}^{M} V_i V_i^H$$

$$S_M \quad W_M$$
ARRAY PROCESSING METHODS (14)

EIGENVALUES OF $R$

\[ \sigma^2 \rightarrow \lambda_3 + \sigma^2 \]

\[ \rho_1 \]

\[ S+N \]

\[ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad I \quad I+1 \]

\[ M \]

\[ N \]
ARRAY PROCESSING METHODS (15)

**SIGNAL SUBSPACE**

\[ \{ V_1, V_2, \ldots, V_I \} \]

\[ \{ \rho_1, \rho_2, \ldots, \rho_I \} \]

- CONVENTIONAL
- ML
- AR
- THERMAL NOISE

**NOISE SUBSPACE**

\[ \{ V_{I+1}, V_{I+2}, \ldots, V_M \} \]

\[ \{ \rho_{I+1}, \rho_{I+2}, \ldots, \rho_M \} \]

- MUSIC
- EIGENVECTOR
- PISARENKO
ARRAY PROCESSING METHODS (16)

SIGNAL SUBSPACE METHODS

- RECONSTRUCT THE "NOISE FREE" SPATIAL COVARIANCE MATRIX

\[
\tilde{R} = \sum_{i=1}^{\text{I}} \rho_i V_i V_i^H \quad \text{and} \quad \tilde{R}^{-1} = \sum_{i=1}^{\text{I}} \frac{1}{\rho_i} V_i V_i^H
\]

- METHODS

**CONV:** \( P(\theta) = c^H \tilde{R} c \)

**ML:** \( P(\theta) = \frac{1}{c^H \tilde{R}^{-1} c} \)

**AR:** \( P(\theta) = \frac{1}{|u^T \tilde{R}^{-1} c|^2} \)

**TH.N:** \( P(\theta) = \frac{1}{|u^T \tilde{R}^{-H} \tilde{R}^{-1} c|^2} \)
ARRAY PROCESSING METHODS (17)

EXAMPLE Sources at: -54°, -42°, 0°, 15°, 22°, 30°,
ARRAY PROCESSING METHODS (18)

- **NOISE SUBSPACE METHODS**

  - **Basic estimator**

  $$P(\theta) = \frac{1}{\sum_{i=I+1}^{M} q_i |\mathbf{c}^H(\theta) \mathbf{v}_i|^2}, \quad -90^\circ \leq \theta \leq 90^\circ$$

  where $\mathbf{c}(\theta)$ is the steering vector

  - **MUSIC:** $q_i = 1$ for all $\{i\}$

  - **EIGENVECTOR:** $q_i = 1/\rho_i$

  - **PISARENKO:** $q_{I+1} = \ldots = q_{M-1} = 0$, $q_M = 1$
ARRAY PROCESSING METHODS (19)

- USING SVD

- Obtain SVD of covariance matrix

\[
R = \sum_{i=1}^{M} \rho_i V_i S_i^H = \sum_{i=1}^{I} (\lambda_i + \sigma^2) V_i S_i^H + \sum_{i=I+1}^{M} \sigma^2 V_i S_i^H, \quad \rho_1 \geq \rho_2 \geq \ldots \geq \rho_M
\]

signal + noise

noise

- SIGNAL SUBSPACE METHODS

\[
\tilde{R} = \sum_{i=1}^{I} \rho_i V_i S_i^H, \quad \tilde{R}^{-1} = \sum_{i=1}^{I} \frac{1}{\rho_i} S_i V_i^H
\]

MxM

MxM

- NOISE SUBSPACE METHODS

\[
\hat{P}(\theta) = \frac{1}{\sum_{i=I+1}^{M} q_i \left[ c^H(\theta) V_i S_i^H c(\theta) \right]}
\]
ARRAY PROCESSING METHODS (20)

USING SVD
1) Obtain SVD of $M \times M$ covariance matrix $R$
2) $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_M$

In practice it might not be easy to distinguish noise from signal subspace as above
3) Minimum description length (MDL) criterion

$$MDL(i) = -\log \left( \frac{\prod_{k=i+1}^{M} \rho_k}{\left( \frac{1}{M-i} \sum_{k=i+1}^{M} \rho_k \right)^{M-i}} \right)^N + \frac{1}{2} i (2M - i) \log N$$

$i = 0, 1, \ldots, M-1$

4) Pick $I$: $\text{MDL}_{\min}(i) = \text{MDL}(I-1)$
5) if $I > (M-1)$ use signal subspace methods
RESOLVING COHERENT SOURCES OR TARGETS (1)

Very difficult problem: Resolve fixed-phase coherent sources (RF in radar) which are spatially separated by less than the beamwidth of the array sampling aperture.

If the coherent sources maintain their fixed-phase relationship and if the array elements do not move then the signal covariance matrix has one unique eigenvalue:

\[ S_M = A S A^H, \quad \text{Rank}[S_M] = 1 = 1 \]

All coherent sources map into one eigenvalue.

\( S \) is non-diagonal and singular.

- \( S_M \) is non-Toeplitz (spatial signal is not "stationary")
RESOLVING COHERENT SOURCES OR TARGETS (2)

Example (1): M=8 element array with half-wavelength spacing, two equal-strength 30dB coherent sources located at 16° and 24° with fixed-phase difference, N=1024 snapshots.

Example (2): in radar the direct and specular component behave like coherent sources.
RESOLVING COHERENT SOURCES OR TARGETS (3)

SPATIAL SMOOTHING METHOD

1) Divide linear array into overlapping subarrays

Subarrays:  
\[ s(1): \{1,2,\ldots,p\} \]
\[ s(2): \{2,3,\ldots,p+1\} \]
\[ \vdots \]
\[ s(K)=s(M-p+1): \{M-P+1,\ldots,M\} \]
RESOLVING COHERENT SOURCES OR TARGETS (4)

2) Estimate the spatial covariance matrix of each subarray

\[ R_i : \quad i = 1, 2, \ldots, K \quad \text{where} \quad R_i = E\{Y_i(n)Y_i^H(n)\} \]

\[
p \times p \]

\[ Y_i(n) \] is the vector of received signals at the \( i^{th} \) subarray

Thus:

\[
R_i = A \left( D^{(i-1)} S \left[ D^{(i-1)} \right]^H \right) A^H + W_i
\]

\[
p \times p \quad p \times I \quad I \times I \quad I \times I \quad I \times I \quad I \times p \quad p \times p
\]

\[ D = \text{diag} \left[ e^{-j \omega_0 \tau_1}, \ldots, e^{-j \omega_0 \tau_I} \right] \quad \text{where} \quad \tau_i = \frac{d}{c} \sin \theta_i \]
RESOLVING COHERENT SOURCES OR TARGETS (5)

3) Average the subarray covariance matrices

\[ \bar{R} = \frac{1}{K} \sum_{i=1}^{K} R_i \]

It follows that:

\[ \bar{R} - A \bar{S} A^H + \sigma^2 I_p \]
\[ \bar{S} = \frac{1}{K} \sum_{i=1}^{K} D^{(i-1)} S D^{(i-1)^H} \]

\( \bar{S} \) is nonsingular regardless of the coherence of the signals

CONCLUSION: In resolving coherent sources, apply "incoherent sources techiques" on \( \bar{R} \)

LIMITATION:

1) Resolution is smaller

2) The number of sources that can be detected is less than p
RESOLVING COHERENT SOURCES OR TARGETS (6)

Example: $\text{SNR}=3\,\text{dB}$, $N=600$, coherent sources at $70^\circ$, $85^\circ$, incoherent sources at $130^\circ$
RESOLVING COHERENT SOURCES OR TARGETS (7)

THE FORWARD - BACKWARD LEAST SQUARES AR METHOD

Given the data array

\[ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad \ldots \quad M-1 \quad M \]

Apply the FBLS AR method on each one of the snapshots independently. We may also use the burg technique, CLS method etc.

- Good performance if model order is accurately estimated.

- Modified FBLS (Kumaresan, Tufts)
**Definition:** A beamformer is a processor that in conjunction with an array of sensors provides a form of spatial filtering.

**Objective:** Given a # of desired signals and a # of interferers (incoherent from the signals), then reject the interferers.

*Case I:* D.O.A. for desired signal is known

*Case II:* D.O.A. for the desired signal is not known

**Implementation:** Shape the sensor array pattern by appropriately weighting the sensor outputs so that maximum gain is placed at the direction of "desired signal" and minimum gain (nulls) at the direction of interferers.
BEAMFORMING (2)

BEAMFORMER CONFIGURATION (Narrowband)

Incident wave

\[ x_1(n) \]

Quadrature Hybrid

\[ y_1(n) \]

\[ w_1^* \]

\[ x_2(n) \]

Quadrature Hybrid

\[ y_2(n) \]

\[ w_2^* \]

\[ x_M(n) \]

Quadrature Hybrid

\[ y_M(n) \]

\[ w_M^* \]

error \[ e(n) \]

desired signal \[ d(n) \]

\[ A_i \cos [\omega_0 n + \phi] \]

\[ A_i \sin [\omega_0 n + \phi] \]

Quadrature Hybrid

\[ w_{iR} \]

\[ w_{iI} \]

\[ A_i \cos [\omega_0 n + \phi] + j A_i \sin [\omega_0 n + \phi] \]

Analytic Form

\[ w_i^* \]

\[ e(n) \] (complex)

\[ d(n) \] (real)
BEAMFORMING (3)

BEAMFORMER CONFIGURATION (Wideband)

Tapped delay line $i$

\[ z^{-1} \]
BEAMFORMING (4)

SOLUTION

MMSE solution:

Minimize: $E\{|e(n)|^2\}$  \textit{w.r.t.} $\{w_{i,k}\}$

Solution:

$$W_{\text{opt}} = R_y^{-1} \cdot r_{yd}$$

where: $R_y = \frac{1}{N} \sum_{n=1}^{N} Y(n)Y^H(n)$ and $r_{yd} = \frac{1}{N} \sum_{n=1}^{N} Y(n)d^*(n)$

$Y(n) = [y_1(n), y_1(n-1),..., y_1(n-L), y_2(n),..., y_2(n-L),..., y_M(n),..., y_M(n-L)]^T$

$W_{\text{opt}} = [w_{1,0},..., w_{1,L}, w_{2,L},..., w_{2,L},..., w_{M,1},..., w_{M,L}]^T$
BEAMFORMING (5)

- **ARRAY PATTERN** (Narrowband)

Assuming reference node on top

Array pattern:

\[
F(\theta, \lambda) = \sum_{i=1}^{M} w_i^* e^{-j \frac{2\pi d}{\lambda} (i-1) \sin \theta}
\]

or

\[
F(\theta, \lambda) = W^H \cdot \varphi(\theta, \lambda)
\]

\( \varphi(\theta, \lambda) \): steering vector at wave \( \lambda \)
LINEAR CONSTRAINED MINIMUM VARIANCE BEAMFORMING (LCMV)

Given the spatial covariance matrix \( R_y = \frac{1}{N} \sum_{n=1}^{N} Y(n) Y^H(n) \)

Minimize \([W^H R_y W]\), s.t.c. \(W^H \zeta(\theta_d, \lambda_d) = 1\), w.r.t. \(\{w_i\}, i = 1, \ldots, M\)

where: \(\theta_d, \lambda_d\): D.O.A., wavelength of desired signal respectively

SOLUTION

\[ W = \frac{R_y^{-1} \zeta(\theta, \lambda)}{\zeta^H(\theta, \lambda) R_y^{-1} \zeta(\theta, \lambda)} \]
ADAPTIVE BEAMFORMING (1)

LINEAR ADAPTIVE ARRAY (1)

Note:

- Use LMS, RLS or ... adaption rules
- Reference (desired) signal must be known
- DOA for the desired signal is not known
- M sensor can be rejected up to M-1 interferers
ADAPTIVE BEAMFORMING (2)

LINEAR ADAPTIVE ARRAY (2) (3 sensors)

- LMS ALGORITHM
  1) Initialization: \( \mathbf{W}(0) = 0 \)
  2) Update equation: \( \mathbf{W}(n) = \mathbf{W}(n-1) + \mu \mathbf{X}(n-1) e(n-1) \)
     where \( e(n) = d(n) - \mathbf{W}^H(n) \mathbf{X}(n) \)
  3) \( 0 < \mu < \frac{2}{\mathbf{X}^H(n) \mathbf{X}(n)} \)

where \( d(n), e(n) \) are real
\[
\mathbf{W}(n) = [w_{1,1}(n), w_{1,2}(n), w_{2,1}(n), w_{2,2}(n), ..., w_{M,1}(n), w_{M,2}(n)]^T
\]
\[
\mathbf{X}(n) = [y_{1,1}(n), y_{1,2}(n), y_{2,1}(n), y_{2,2}(n), ..., y_{M,1}(n), y_{M,2}(n)]^T
\]

4) \( F_n(\theta, \lambda) = \sum_{i=1}^{M} (w_{i,1} - jw_{i,2}) e^{-j\frac{2\pi d}{\lambda}(i-1)\sin\theta} \)
ADAPTIVE BEAMFORMING (3)

ADAPTIVE SIDELOBE CANCELLER (1) (Narrowband)

- Beamforming network forms a set of orthogonal beams
- Consider as reference sensor the one in the middle
- Assume D.O.A. for desired signal is known
- M sensors can reject up to M-2 interferers

5 sensors
4 orthogonal beams
ADAPTIVE BEAMFORMING (4)

ADAPTIVE SIDELOBE CANCELLER (2)

- Assume for the moment that outputs of the sensors are equally weighted and have a uniform phase. Then,

Response of array to an incident phase $\Phi$

$$A(\theta, \alpha) = \frac{M-1}{2} \sum_{n=-\frac{M-1}{2}}^{\frac{M-1}{2}} e^{jn\phi} e^{-jn\alpha} = \frac{\sin\left[\frac{1}{2} (M+1)(\phi - \alpha)\right]}{\sin\left[\frac{1}{2} (\phi - \alpha)\right]}$$

for $\alpha = \frac{\pi}{M} k$, $k = \pm 1, \pm 3, ..., \pm M-2$

M-1 orthogonal array beams can be generated

- $\phi = \pm \frac{2\pi d}{\lambda} \sin\theta$, special case ($d = \frac{\lambda}{2}$, $-\pi \leq \phi \leq \pi$, $-90^\circ \leq \theta \leq 90^\circ$)
ADAPTIVE BEAMFORMING (5)

■ ADAPTIVE SIDELOBE CANCELLER (3)

1) Given M sensors, form M-1 orthogonal beams by combining appropriately the sensor outputs.

2) Place one of the beams at the direction of the desired signal. Then the other auxiliary beams will place a null at that direction.

3) Weight each beam with the weights \( \{w_i(n)\} \), \( i = 1, \ldots, M \). Due to the symmetry of the problem the weights are real.

4) Update \( \{w_i(n)\} \), \( i = 1, \ldots, M \) using LMS algorithm.

5) The array beam at instant \( n \) is

\[
F_n(\theta, \lambda) = \sum_{i=1}^{M-1} w_i(n) \frac{\sin\left[\frac{1}{2}(M+1)(\phi - \alpha_i)\right]}{\sin\left[\frac{1}{2}(\phi - \alpha_i)\right]}
\]
Example: INTERFERENCE REJECTION (1)

Given an adaptive array with M=3 elements uniformly spaced with d=1 (λ_d = 2d). The data received by the array are described by the equation

\[ x_m(n) = \cos[2 \pi f_d (n-1) + \phi_1] + I \cos[2 \pi f_I (n-1) + \phi_2] + w(n) \]

\( m=1,2,3; \quad n=1,2,...,512 \)

where:

- \( f_d \) is the frequency of desired signal
- \( f_I \) is the frequency of an interferer
- \( w(n) \) if AWGN
ADAPTIVE BEAMFORMING (7)

Example: INTERFERENCE REJECTION (2)